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LOGICS, INTERPRETATIONS AND INFORMAL RIGOR

LÓGICAS, INTERPRETAÇÕES E RIGOR INFORMAL

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ABSTRACT *The development of non-classical logics brought to light the question of their philosophical interpretation. Given a non-classical logic* ***L****, what is the informal/philosophical interpretation of its vocabulary? We find in the literature arguments defending that logical systems do not have a canonical philosophical interpretation, so the same logical system* ***L*** *has different philosophical interpretations. Although the thesis that a logic can be philosophically interpreted in different ways is well known, the thesis that a specific philosophical interpretation can be captured by different logics has not been widely explored. In this paper, we argue that if Kreisel's informal rigor method is adequate to show that formal notions of logical consequence capture informal notions of logical consequence, then a specific philosophical interpretation only corresponds to a unique logic.*

Keywords: *Non-classical logics. Philosophical interpretation. Informal rigor.*

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RESUMO *O desenvolvimento das lógicas não-clássicas trouxe à luz a questão da sua interpretação filosófica. Dada uma lógica não-clássica* **L***, qual é a interpretação informal/filosófica do seu vocabulário? Encontramos na literatura argumentos defendendo que os sistemas lógicos não possuem uma interpretação filosófica canônica. Desse modo, um mesmo sistema lógico possui diferentes interpretações filosóficas. Embora a tese de que uma lógica pode ser interpretada filosoficamente de diferentes maneiras seja bem conhecida, a tese de que uma interpretação filosófica específica pode ser capturada por diferentes lógicas não foi amplamente explorada. Neste artigo, argumentamos que, se o método do rigor informal de Kreisel é adequado para mostrar que as noções formais de consequência lógica capturam noções informais de consequência lógica, então tais noções informais correspondem apenas a uma lógica única.*

Palavras-chave: *Lógicas não clássicas. Interpretações filosóficas. Rigor informal.*

1. Introduction

The development of non-classical logics brought to the light the question of their informal interpretation. Given a non-classical logic **L**, what is the informal interpretation of its vocabulary? In particular, paraconsistent logics, capable of controlling contradictions, played an important role in a more recent discussion of this question. In a series of papers, Carnielli and Rodrigues (2015, 2019a, b, c, 2022) argue that paraconsistent logics must be interpreted epistemically, contrary to the ontological interpretation proposed by Priest (1979, 2006).[1] According to this epistemic interpretation, a contradiction *A* & non-*A* expresses the existence of contradictory evidence about an *A* proposition. In order to formalize this epistemic interpretation, the authors propose two logical systems: the *Basic Logic of Evidence* and *The logic of evidence and truth.*

The thesis defended by Carnielli and Rodrigues has been widely discussed in the literature. Barrio (2018) and by Barrio and Da Ré (2018) argue that a logical system **L** does not have a canonical philosophical interpretation, so that the same logical system **L** has different philosophical interpretations. Bezerra and Venturi (2021) generalize Barrio's (2018) objection, showing that the *informal*

1 It is important to say that Priest does not defend that paraconsistent logics force us to an ontological view that asserts the existence of contradictions in reality, *dialetheism*. He defends that a paraconsistent logic is adequate to formalize a context where dialetheism is true (Priest, 2019).

rigor method proposed by Kreisel (1967) can be used to show that the formal notions of validity of **L** cannot capture a unique notion of informal validity. Arenhart (2022) argues that both Barrio and Barrio and Da Ré's objections can be recasted only in terms of Priest's distinction between pure and applied logic (Priest, 2005).[2] Therefore, these objections show that a formal system does not force a unique interpretation or that it can be applied only in one context.

Although the thesis that a given logic can be interpreted philosophically in different ways is well known, the thesis that a specific philosophical interpretation can be captured by different logics has not been widely explored. In this paper, I argue that if the informal rigor method is adequate to show that formal notions of logical consequence capture informal notions of logical consequence, then such informal notions only correspond to a unique logic.

This paper is organized as follows. In Section 2, we present the current discussion about logics and their interpretations. We will present the view that logics have multiple interpretations and nothing in a formal system forces a privileged interpretation. In Section 3, we present Kreisel's informal rigor and his squeezing argument, showing that the formal notions of validity of First-Order Logic capture the informal notion of truth in all structures. We also show how to extend his argument in order to find other informal notions of validity for classical logic, and we show versions of this argument for intuitionistic logic. In Section 4, we argue that an informal notion of validity can be captured by only one logic. In Section 5, we close the discussion with a few remarks.

2. Logics and their interpretations: one logic, many interpretations

There is a myriad of non-classical logics. Many logics were proposed to formalize concepts that classical logic apparently does not give a good account of. For example, some non-classical logics were proposed to deal with inconsistent reasoning, such as da Costa's hierarchy of paraconsistent logics **Cn** (da Costa, 1974). Many other logics were proposed to deal with semantic paradoxes, such as the *Logic of Paradox* **LP** (Asenjo, 1966; Priest, 1979). Although we will not give an exhaustive list of possible motivations to propose a non-classical logic, it is important to observe that they arise from the need to formalize important concepts of our reasoning.

Although non-classical logics were in general proposed to solve problems that classical logic is not apparently able to give an adequate account, the

2 Barrio et al. (2025) argue that the notion has an important philosophical role and that it cannot be collapsed in to the distinction between pure and applied logics.

informal interpretation of these logics is still an open problem. There is a general agreement that classical logic formalizes general principles of preservation of truth in the mathematical reasoning (Frege, 1956; Kreisel, 1967; Kennedy and Väänänen, 2017). That is, all its axioms and inference rules state general principles of preservation of truth reasoning. What about the other logics? Of course, it is a matter of fact that some logics do formalize other notions than preservation of truth. For example, the *Brouwer-Heyting-Kolmogorov interpretation* (BHK interpretation) captures the meaning of the logical constants of intuitionistic logic **IL**, which allows one to say that **IL** states general deductive principles of constructible reasoning (van Dalen, 1986; van Dalen and Toelstra, 1988). In the case of paraconsistent logics, however, the situation is not easy. As it is widely known, these logics handle contradictions without trivializing the whole theory. Despite this interesting feature of paraconsistent logics, the following question raises: *how does one interpret a logical system that tolerates contradictions?*

If one interprets paraconsistent logics as also dealing with preservation of truth, then formulas of the form $A \,\&\, {\sim}A$ states that the formula A as well as its negation ${\sim}A$ are both true. In Priest's terminology, A is a *dialetheia* (Priest et al., 2018). *Dialetheism* is a philosophical thesis that states the existence of dialetheias. According to Priest (1979, 2006), **LP** formalizes the basic intuitions of dialetheism. Of course, as Priest (2019) himself recognizes, this does not prevent that paraconsistent logics can be interpreted differently. Another possible interpretation for these logics is epistemic. In the literature, we find many proposals to interpret these logics in terms of information.[3] As Carnielli and Rodrigues (2012, 2015, 2019a, b, c, 2022) argue, paraconsistent logics must be seen as preserving the notion of *evidence*. According to them, evidence for a statement A is understood as *reasons for accepting A*. They defend that it is only in the level of evidences and in the scientific level that contradictions exist.[4]

Carnielli and Rodrigues present two logical system that formalize the notion of preservation of evidence: the *basic logic of evidence* (**BLE**) and the *logic of evidence and truth* (**LETJ**) that extends **BLE** with the classicality operator o. Both logics are paraconsistent and paracomplete, because it is possible to have evidence for A and ${\sim}A$ and but no evidence for B, hence invalidating the *rule of*

3 In Pinter (1980), Carnielli et al. (2004), Blasio (2017), Belnap (2019), we find arguments that defend that paraconsistent logics are useful to deal with inconsistent information.

4 There are many textual evidences that attest their claim defending that contradictions are epistemic. In the aforementioned references, the reader will find textual evidence of our claim.

explosion (in symbols: *A, ~A* |-/- *B*); and it is also possible to have no evidence for *A* either *~A*, hence invalidating the *law of excluded middle* (|-/- *A* v *~A*). As Carnielli & Rodrigues show, it is possible to recover classical inferences in **LETJ** by stipulating the classicality of the sentences involved in the inferences. Both logics are presented in terms of natural deduction systems, and we refer the reader to consult these proof systems in their work.

Given that **LETJ** is capable to recover classical inferences, Carnielli and Rodrigues argue that paraconsistent and classical logics can coexist without rivalry. While classical logic deals with preservation of truth and intuitionistic logic deals with preservation of constructive provability, paraconsistent logic deals with preservation of evidence. Since the consequence relation of these logics deal with different notions, there is no rivalry between them. Then, Carnielli and Rodrigues hold a pluralistic attitude about logic in maintaining that different logics formalize different informal notions of validity. We can say that they hold a version of *contextual pluralism* (da Costa, 1980; Caret, 2017), where different logics formalize different deductive contexts. A *context* can be understood as "admissible class of cases that function as logically salient alternatives" (Caret, 2017, pg. 753). Then, different logics formalize different notions of validity.

Carnielli and Rodrigues' proposal of interpreting paraconsistency as preservation of evidence faces some objections. Among these objections, we will focus on the objections that focus on Carnielli and Rodrigues' claim that paraconsistent logics must deal with preservation of evidence. The objections we will present here were formulated by (Barrio, 2018; Barrio and da Ré 2018; Arenhart 2021, 2022).[5]

Barrio (2018) and Barrio and Da Ré (2018) argue that paraconsistency itself is a property of logics that invalidate the rule of explosion and there is nothing in these logics that compels them to be interpreted in alethic terms or epistemologically. Paraconsistent logics, as well as every other logical system, are *pure logics*. Pure logics are formal languages endowed with a consequence relation. At this level, pure logics only say what sentences follow from other ones. These logics can also be *applied* to formalize our ordinary or mathematical reasoning, electric circuits, and so on.[6] From this distinction between pure and applied logics, Barrio and Da Ré argue that applied logics can receive different *philosophical interpretations*, which give us "additional understanding of certain pure logical theories" (Barrio and Da Ré, 2018, pg.

5 These arguments are also recasted in (Bezerra, 2024).

6 The distinction between pure and applied logic can be found in (Priest, 2005).

159). Barrio et al. (2025) defines a philosophical interpretation as a conceptual description of the logical vocabulary of constants of a logic **L** in terms of an informal notion, as well as the notion of logical consequence of **L**. Then, even if a logic **L** is to be applied to a certain context, **L** can be philosophically interpreted in different ways.[7]

Barrio (2018) advances the criticisms of Barrio and da Ré and argues that logics do not have canonical interpretations. That is, a logic **L** does not compels a specific interpretation *i***L** even if **L** was formulated to formalize ***iL***. He argues that it is possible to give an alethic interpretation for **BLE** and **LETJ**, given that **BLE** is equivalent to Nelson's logic **N4**, that is model-theoretically characterized by a four-valued possible worlds semantics, whose truth-values are interpreted as follows: *just true*, *just false*, *both true and false*, *neither true nor false*. This shows that **BLE** can receive both an ontological interpretation and an epistemological one. Nothing in these logics obligate us to interpret them in an exclusive way.[8]

Arenhart (2022) argues that the notion of philosophical interpretation play no explanatory role besides the distinction between pure and applied logics. According to him, once a logic **L** is applied to formalize a certain notion, the concept of philosophical interpretation plays no significant role, because the application itself of **L** gives meaning to the logical constants of this logic. He argues that the notion of applied logics is enough to undermine Carnielli and Rodrigues' claim that paraconsistent logics are to be interpreted in terms of evidence. According to him, both **BLE** and **LETJ** are applications of pure logics to formalize the concept of preservation of evidence. But nothing in these logics prevent that they can be applied to formalize other notions of consequence.[9]

The idea that the same logic can receive more than one interpretation is well accepted in the literature. Rodrigues and Carnielli (2022) agree that **BLE** and **LETJ** can receive other interpretations or that they can be applied to more

7 In (Barrio et al., 2025), we find examples in favor of the proper distinction between philosophical interpretations and applications of a logic. The non-transitive logic **ST** (Cobreros et al. 2012; Ripley, 2013) is an example of logic that is usually applied in the context of semantic paradoxes, but there is no consensus on how to interpret its logical vocabulary.

8 In their papers, Carnielli and Rodrigues recognize that **BLE** and **LETJ** can receive an alethic interpretation. Their main point is that the junction of these logics with whe evidence interpretation is anti-dialetheist. Although we will not evaluate this point, it is clear that this recognition shows the soundness of Barrio's and Barrio and Da Ré's objections.

9 There are other objections to Carnielli and Rodrigues' approach to paraconsistency. We refer the reader to Lo Guercio and Szmuc (2018) and Arenhart (2021) that argue that **BLE** and **LETJ** fail to capture the notion of preservation of evidence. In (Rodrigues and Carnielli, 2022), we find responses to each objection to **BLE** and **LETJ** presented here.

than one context. Thus, there is a pluralism with respect to interpretations/applications of logics. Before we advance to our point, it is important to take a position with respect to the notion of philosophical interpretation. Here we will follow Barrio (2018) and Barrio and Da Ré (2018)'s terminology and we will keep reaffirm the importance of the notion of philosophical interpretation in this debate. Even if we will not develop much this point, we agree with Rodrigues and Carnielli's claim that the interpretation of a logical system is an intermediary step between a pure logic and its application. More recently, Barrio et al. (2025) argue that philosophical interpretation must be properly distinguished from applications. Then, from now on, we will keep referring to philosophical interpretation instead of applications.

Now we raise the following question: can the same philosophical interpretation be captured by different logics? In what follows, we defend that different logics capture different philosophical interpretations. In order to defend our position, we present Kreisel's argument, nowadays known as *squeezing argument*, that shows that classical First-Order Logic captures the informal concept of validity as *truth in all structures*. We show that this argument can be extended to show that the formal notions of consequence of a logic **L** cannot capture a unique informal notion of validity. Moreover, we show that, by the same argument, distinct logics capture distinct notions of informal validity.

3. Informal rigor

In this Section, we argue that Kreisel (1967)'s method of *informal rigor* can be used to show that the formal notions of consequence of a logic **L**, semantic and syntactic, cannot capture a single informal notion of validity. According to Kreisel, informal rigor is an activity of conceptual analysis of intuitive notions to "eliminate the doubtful properties of the intuitive notions when drawing conclusions about them" (Kreisel, 1967, pg. 138). Logical validity was one of the concepts that he analyzed. Besides the formal notions of validity, semantic and syntactic, Kreisel defends that there is an informal notion of validity that is irreducible to both formal notions. Now, we present his argument. Let A be a formula of First-Order Logic (**FOL**). Then, he presents the following three notions of validity:

(Informal) Val(A): A is true in all structures.

(Semantic) V(A): A is true in all structures in the cumulative hierarchy.

(Syntactic) D(A): A is derivable by means of some fixed set of formal rules.

According to Kreisel, both V and D are *formal* notions, because both are presented in well-structured conceptual frameworks. As he argues, the *informality* of Val lies in the non-specification of the size of the domain of the structures. Because Val is informal, it does not make sense to present a formal proof relating these concepts. What we can do, at most, is to argue in favor of the relation of these concepts. Even so, Val is irreducible to both V and D. According to him, Val is irreducible to D because nobody reasons with formal rules. On the other hand, it is expected that the proof systems of **FOL** are *informally sound*, in the sense that all provable formulas A are valid in the sense of Val, and the rules of the deductive system preserve informal validity. So,

$$\text{(A) } \mathrm{D}(A) \Rightarrow \mathrm{Val}(A).$$ [10]

Val is also irreducible to V, because Val comprehends structures whose domains are class sized, whereas the semantic structures of V are set sized. On the other hand, if A is valid in all structures, A is valid in all structures whose domain is set sized. Then,

$$\text{(B) } \mathrm{Val}(A) \Rightarrow \mathrm{V}(A).$$

As we know, **FOL** has a completeness theorem. Then, every valid formula is provable. Then:

$$\mathrm{V}(A) \Rightarrow \mathrm{D}(A).$$

Then:

***Squeezing Argument* (*SA*).** Kreisel *squeezing argument* runs as follows:

1.	$\mathrm{D}(A) \Rightarrow \mathrm{Val}(A)$	(A)
2.	$\mathrm{Val}(A) \Rightarrow \mathrm{V}(A)$	(B)
3.	$\mathrm{V}(A) \Rightarrow \mathrm{D}(A)$	Completeness theorem
4.	$\mathrm{V}(A) \Rightarrow \mathrm{Val}(A)$	From 1,3
5.	$\mathrm{V}(A) \Leftrightarrow \mathrm{Val}(A) \Leftrightarrow \mathrm{D}(A)$	From 2,4

10 In this case, one can show, by means of an informal justification, that all axioms of **FOL** are valid in the sense of Val, and that the inference rules preserve Val. Such an informal justification is similar to the justification we find in the intuitionistic logic, where Brouwer-Heyting-Kolmogorov interpretation is usually taken to be the interpretation of the intuitionistic constants (van Dalen, 1986). In subsection 3.1, we will consider the intuitionistic case.

According to Andrade-Lotero and Novaes (2012), the ***SA*** gives a philosophical understanding to the completeness theorem because it shows that there is an informal concept of validity linking the two formal ones. It is important to observe that this argument only works for logics that are semantically complete.[11] Another thing that it is important to observe is that the informal notion Val is a highly theorized concept of logical validity. So, one might object that Val does not correspond to our intuitive/pre- theoretical concept of validity. To counter this objection, it suffices to observe that we do not even know what intuitive validity is. Indeed, there is a widespread suspicion about such an intuitive concept of validity.[12] As Kennedy and Väänänen (2017) observe, Val captures a notion of validity present in mathematical practice that is, according to them, semantic.

As one can see, ***SA*** is quite simple.[13] Because of that, different versions of this argument were formulated in the literature in order to find notions of validity that are closer to the natural language, as well as versions of this argument to other logics. As an example of the first case, Shapiro (2005) formulates variants of ***SA*** for more informal notions of validity. Let A^{ln} and $\sum^{ln}$ be the counterparts of the sentence A and the set of sentences $\sum$ in the natural language, respectively. Shapiro defines the following notions of consequence:

> $\mathrm{Val_B}(\sum^{ln}, A^{ln})$: A^{ln} is logical consequence of $\sum^{ln}$ in the *blended* sense; that is, it is not possible to every member of $\sum^{ln}$ to be true and A^{ln} be false, and this impossibility holds in virtue of the meaning of the logical terms. $\mathrm{Val_B}(A^{ln})$ means that A^{ln} is valid in the blended sense.
>
> $\mathrm{Val_{Ded}}(\sum^{\mathrm{ln}}, A^{ln})$: A^{ln} is logical consequence of $\sum^{ln}$ in the *deductive* sense; that is, there is a deduction of A^{ln} from $\sum^{ln}$ by a chain of legitimate (gap-free) rules of inference. $\mathrm{Val_{Ded}}(A^{ln})$ means that A^{ln} is valid in the deductive sense.

Shapiro argues that these informal notions capture the aspects of formality and necessity of logical consequence. Arguably, it is obvious that every theorem A of **FOL** is also valid in $\mathrm{Val_B}(A^{ln})$/ $\mathrm{Val_{Ded}}(A^{ln})$ sense. Then, we say that **FOL** is *faithful* with respect to these informal notions of validity. Moreover, if A^{ln} is valid in the sense of $\mathrm{Val_B}$/$\mathrm{Val_{Ded}}$, then A is valid in the sense of V, given that

11 We will say more about incomplete logics in the last section of this paper.

12 We refer the reader to (Smith, 2011; Halbach, 2020; Glanzberg, 2015 Bezerra, 2023) for this discussion. Since Val, as well as the others informal notions we will present here, is highly theorized we prefer to call it informal instead of intuitive. This latter notion seems to be more basic, and less theorized, than the former.

13 Besides its simplicity, its structure requires that the informal notion at issue should be sufficiently sharp in order to be related to the formal notions of validity. Such a sharpening intends to remove ambiguity of the informal notion. For this reason, we think that the informal rigor is an adequate method for relating informal notions to formal ones.

A is a formalization of A^{ln}. So, we say that **FOL** is *adequate* to both informal notions of validity. As Shapiro shows, it is possible to formulate variants of ***SA*** using the notions $\mathrm{Val}_{\mathrm{B}}$ and $\mathrm{Val}_{\mathrm{Ded}}$. Although these notions collapse in the first-order case, they are intentionally different. If A^{ln} corresponds to a second-order sentence, these informal notions of validity do not collapse, because Second-Order Logic is not complete.

The formal notions of validity of **FOL** capture at least three different informal notions: Val, $\mathrm{Val}_{\mathrm{B}}$ and $\mathrm{Val}_{\mathrm{Ded}}$. From a philosophical point of view, these three informal notions are equally legitimate. Then, this shows already in the classical case that **FOL** do not capture a unique informal notion of consequence.

3.1. Squeezing arguments and non-classical logics

Bezerra and Venturi (2021) shows that it is possible to formulate versions of squeezing arguments for non-classical logics. In their paper, they formulate squeezing arguments for **IL** relating the formal notions $V_{\mathbf{IL}}$ and $D_{\mathbf{IL}}$ of intuitionistic logic to some informal notions. First, let A be a formula of **IL**. So:

$$\mathrm{Val}_{\mathbf{I}}(A)\text{: } A \text{ is constructively provable.}$$

It is easy to see that this informal notion of validity is derived from BHK interpretation of intuitionistic logic (van Dalen, 1986). Then, we have that:

$$\text{(A')}\ D_{\mathbf{IL}}(A) \Rightarrow \mathrm{Val}_{\mathbf{I}}(A).$$

Second, if A is constructively provable, then A is also valid in structures whose internal structure is intuitionistic logic. So:

$$\text{(B')}\ \mathrm{Val}_{\mathbf{I}}(A) \Rightarrow V_{\mathbf{IL}}(A).$$

Last, **IL** has a completeness theorem. Then:

$$\mathrm{Val}_{\mathbf{I}}(A) \Rightarrow \mathrm{Val}_{\mathbf{IL}}(A).$$

Given these three implications, Bezerra and Venturi (2021) formulate a ***SA*** for **IL**. As they argue, it is possible to formulate other versions of this argument for **IL** by using the translations of this logic into modal logics. As it is widely known, **IL** is translatable into the modal logic **S4**, using Tarski-McKinsey's translation. This is guaranteed by the following result:

Theorem. Let L and $L_{\Box}$ be the languages of **IL** and **S4**, respectively. There is a translation
$t: L \rightarrow L_{\Box}$ such that **IL** |- A if and only if **S4** |- $t(A)$.

By using translation t Bezerra and Venturi formulate variants of ***SA*** for **IL** with the following informal notions:

$\mathrm{Val}_{\mathbf{I}}+(A)$: t($A$) is informally provable.[14]
$\mathrm{Val}_{\mathbf{I}}*(A)$: t($A$) is known*.[15]

Then, as happens with **FOL**, the formal notions of validity of **IL** are not able to single out a unique informal notion of validity. One could formulate, of course, versions of ***SA*** for the logics **BLE** and $\mathbf{LET_J}$, by adapting Barrio's objections in terms of informal notions of validity. In a more general perspective, arguments like ***SA*** can be used to show that the formal notions of consequence are not able to capture a unique notion of informal validity.

The way that ***SA*** is structured allows us to say that it is an adequate way to test whether an informal notion of validity is compatible with a formal logic **L**. This kind of argument is also adequate to test if a philosophical interpretation is compatible with a logic **L**. The reason is simple: a philosophical interpretation i_C for a logic **L** gives meaning to the logical constants of **L** in terms of a concept c. For example, the standard philosophical interpretation of **FOL** is alethic and the fundamental concept is truth. Now, given the philosophical interpretation of the logical constants of **L**, the definition of informal validity in terms of this philosophical interpretation is straightforward. Then, ***SA*** also shows that the same logic can have different philosophical interpretations. Now, we will argue that ***SA*** shows that a particular notion of informal validity can only be captured by one logic.

4. One interpretation, one logic.

There are many informal notions of validity, and they are equally legitimate from a philosophical point of view. In order to name a few:

14 See Halldén (1949), Burgess (1999) and Dean (2014) for arguments that the modal operator $\Box$ of **S4** can be interpreted as informal provability.

15 According to Stalnaker (2006), S4 captures knowability without doxastic elements. It is clear that this informal notion of knowability is quite idealized. It is a commonsense in the literature that the notion of knowledge is not closed under logical consequence.

I) Validity as preservation of truth.
II) Validity as preservation of constructive proof.
III) Validity as preservation of informal provability.
IV) Validity as preservation of knowability.
V) Validity as preservation of evidence.
VI) Validity from a bilateralist interpretation: *A* follows from Γ if and only if it is not possible to strictly assert all members of Γ and tolerantly deny *A* (Ripley, 2013).

This list does not intend to be exhaustive. Each informal notion of validity is a product of a possible philosophical interpretation of the logical vocabulary. As argued in the previous sections, the same logic can capture more than one informal notion of validity. **IL**, for example, captures II, III and IV, where these two latter notions were obtained by exploiting the translation between **IL** and **S4**. However, the converse does not seem to be true. That is, each notion of informal validity is captured by one logic.

At this point, one could object us by pointing that **IL** and **S4** capture the notions II-IV. We respond this objection by noting that these systems capture these informal notions because **IL** can be translated into **S4**. It is interesting to keep in mind that this modal logic was originally used as an interpretation of intuitionistic propositional calculus (Gödel, 1986). The notions Val_{I+} and Val_{I*} witness this, where the translation *t* is explicitly used. Then, two logics capture the same informal interpretation if there is a translation between them that allows a variant of **Theorem**.

In general, different logics capture different informal notions. The main reason for this asymmetry is due to the fact that logics, once interpreted, are normative with respect to their corresponding interpretations.[16] If there is a squeezing argument relating the formal notions V**L** and D**L** to an informal notion Val***i***, we can say that **L** is normative with respect to Val***i***, because the valid inferences and the theorems of **L** establish norms to reason according to this informal notion. In this perspective, **L** is normative "for some epistemically relevant attitudes directed towards the constituents of the arguments assessed" (Tajer and Fiore, 2022 pg. 228). That is, the validities of **L** will regulate our reasoning about the notion *i* that gives Val_i.

16 There is a well-known discussion about the normativity of logic. For reasons of space, we will not present it here. Instead, we invite the reader to (Russell, 2020; Stei, 2020; Buacar, 2021; Massolo, 2023; Tajer, 2024) for a qualified discussion.

Let i be an informal interpretation of the logical constants and Val_i its corresponding notion of informal validity and $\mathbf{L_1}$ and $\mathbf{L_2}$ two logics such that, for some A, $\mathbf{L_1}$ |-A and $\mathbf{L_2}$ |-/-A. If there is a ***SA*** relating $V_{\mathbf{L1}}$ and $D_{\mathbf{L1}}$ to Val_i, then we have that:

$$V_{\mathbf{L1}}(A) \Leftrightarrow \text{Val}_i(A) \Leftrightarrow D_{\mathbf{L1}}(A).$$

Since $\mathbf{L_1}$ and $\mathbf{L_2}$ disagree with respect to A, a ***SA*** relating $V_{\mathbf{L1}}$ and $D_{\mathbf{L1}}$ to Val_i will not work, since the formula A is valid with respect to Val_i. This is close to what da Costa (1980) call *uniqueness* of logic in rational contexts. According to him, given a rational context c, there is one logic that regulates it. This is expressed in the following passage:

> Metaphorically speaking, this second principle assures us, that once the rules of the game are fixed, they must not be altered. A change would immediately modify the initial game, transforming it into another one. More exactly, modifying the underlying logic of a rational context would convert it into a different context. (da Costa, 1980, pg. 47)

Rodrigues and Carnielli (2022) also argue that a single interpretation cannot be captured by different logics. They give the example of Heyting's logic **H** and Johansson's logic **J**. Both logics were designed to formalize intuitionistic reasoning, but **H** and **J** differ on their validities. This means that they capture different informal notions of constructive proof, because **H** will have at least one theorem that **J** does not. Then, we cannot run ***SA*** for both logics using the same informal notion of validity.[17]

If two logics $\mathbf{L_1}$ and $\mathbf{L_2}$ are said to capture a philosophical interpretation i, we can say that $\mathbf{L_1}$ and $\mathbf{L_2}$ dispute such a notion. In the case of truth, we cannot say that **FOL** and **LP** or **IL** capture the same notion of truth, because one is paraconsistent and the other is paracomplete. In this case, we can say that there is a dispute between what formalizes correctly the notion of preservation of truth, not that these logics capture the same notion.

In this paper, although we discussed the relation between **IL** and **S4**, we did not analyzed versions of ***SA*** for modal logics. The reason is that status of logicality of modal operators is an open problem in the literature.[18] Even so, we

17 Rodrigues and Carnielli understand a philosophical interpretation of a logic **L** as the intended meaning to the constants motivated by a philosophical concept. This characterization is precise enough to see that two distinct system cannot share the same philosophical interpretation. Perhaps, one might say that such a understanding implies that the disagreement between two logics may rest upon a verbal dispute. Here we will not take an instance of this discussion, but we point that this is also a consequence of the plurality of logics and it is by no means a negative thing.

18 We invite the reader to read Dutilh Novaes (2014) for a qualified discussion about logicality of modal operators. In her characterization of logicality, modal operators can be taken as logical constants. But here we prefer to not analyse these logics by means of ***SA*** in order to avoid to take any position about this discussion.

present an example from modal logics to defend our view about logics and their informal interpretations. Take, for example, the modal logic **GL** (Boolos, 1995), whose operator $\Box$ is usually interpreted as *provability in Peano Arithmetic*. Indeed, it is possible to prove that it is actually the case (Solovay, 1963). On the other hand, $\Box$ can be interpreted in **S4** as *informal provability*, which is not particular to a formal system. As Gödel (1986) observes, **S4** does not capture arithmetical provability because this logic has the axiom $\Box A \to A$ (T). This axiom, as it is widely known, is incompatible with arithmetical provability. Thus, this example makes clear that **S4** and **GL** capture different concepts of provability. There are other notions interesting to be analyzed, such as the interpretation of $\Box$ as *knowability*. We cannot say that the logic **S4.2** and an epistemic logic that fails to validate the axiom $\Box(A \to B) \to (\Box A \to \Box B)$ are capturing the same notion of knowledge. These examples manifest this general relation between logics and informal interpretations: although logics can be interpreted in different ways, an informal interpretation can be captured by only one logic.

5. Conclusion

In this paper, we argue that an informal interpretation can be captured by only one logic, although a logic can capture more than one informal interpretation. The formulation of variants of the ***SA*** were useful in showing that an informal notion of validity is related to only one logic. The structure of the argument supposes that logics are normative with respect to the informal notions that they are related. Then, the different validities of these logics mean that they have different norms.

Since we used ***SA*** to relate logics and their respective informal notions, our arguments are restricted to logics that have a completeness theorem. For logics that are incomplete, as it happens with Second-Order Logic (**SOL**), this argument does not work. So, possibly our argument will face some limits. As Kennedy and Väänänen (2017) show, it is possible to formulate ***SA*** for fragments of **SOL**, if one considers Henkin semantics,as well as some extensions of **FOL**. For these restrictions, our arguments will hold.

There are some cases of incomplete logics worth to consider. There are modal logics that are not characterizable by any class of Kripke frames. Even if these logics are less complex than **SOL**, the situation seems to be different with respect to the possibility of formulating a version of ***SA*** for them. For these modal logics, it is not the case. Since the completeness theorem is mandatory for the formulation of Kreisel's argument, the informal rigor faces an obstacle in these cases.

The impossibility of formulating a ***SA*** for these incomplete modal logics may suggest that to the method of informal rigor is not adequate to interpret a logic. However, we think that it is not the case. Since these logics do not have models that validate their theorems, there is no informal notion that bridges both definitions of validity. In this sense, one cannot argue that a particular philosophical interpretation indeed corresponds to the formal notions of validity. As we saw, the conclusions of the ***SA*** presented above are in the form of biconditionals. The philosophical interpretations that these logics may receive are interpretations in a wider sense, because there is no notion bridging the formal notions of validity.

An interesting discussion that can be developed further is the relation of squeezing arguments and logical pluralism. As we argued above, the plurality of informal notions also suggest a form of contextual pluralism, as defended by da Costa (1980) and Caret (2017). Another possible line of research is to relate Bezerra and Venturi (2021)'s view with the interpretational pluralism defended by Tajer and Fiore (2022). This will be investigated in a future work.

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References

ANDRADE-LOTERO, E., DUTILH NOVAES, C. "Validity, the squeezing argument and alternative semantic systems: the case of Aristotelian syllogistic". *Journal of Philosophical Logic*, Vol. 41, Nr. 2, pp. 387-418, 2012.
ARENHART, J. R. B. "The evidence approach to paraconsistency versus the paraconsistent approach to evidence". *Synthese*, Vol. 198, Nr. 12, pp. 11537-11559, 2021.
______. "Interpreting philosophical interpretations of paraconsistency". *Synthese*, Vol. 200, Nr. 6, pp. 449, 2022.
ASENJO, F. G. "A calculus of antinomies". *Notre Dame Journal of Formal Logic*, Vol. 7, Nr. 1, pp. 103-105, 1966.
BARRIO, E., DA RE, B. "Paraconsistency and its philosophical interpretations". *The Australasian Journal of Logic*, Vol. 15, Nr. 2, pp. 151-170, 2018.
BARRIO, E. A., BEZERRA, E., DA RÉ, B. "Philosophical interpretations matter". *Principia: An International Journal of Epistemology*, Vol. 29, Nr. 2, pp. 203-225, 2025.
BARRIO, E. A. "Models & proofs: lfis without a canonical interpretations". *Principia: An International Journal of Epistemology*, Vol. 22, Nr. 1, pp. 87-112, 2018.
BELNAP, N. D. "A useful four-valued logic". In: DUNN, J. M.; EPSTEIN, G. (eds.). *Modern Uses of Multiple-Valued Logic*, pp. 5-37. Springer, 1977.
BEZERRA, E., VENTURI, G. "Squeezing arguments and the plurality of informal notions". *Journal of Applied Logic*, Vol. 8, Nr. 7, pp. 1899-1916, 2021.
BEZERRA, E. "On validity paradoxes and (some of) their solutions". *Principia: An International Journal of Epistemology*, Vol. 27, Nr. 3, pp. 519-538, 2023.
______. "Paraconsistency, evidence and semantic incompleteness". *Análisis Filosófico*, Vol. 44, Nr. 1, pp. 117-140, 2024.
BLASIO, C. "Revisitando a lógica de Dunn-Belnap". *Manuscrito*, Vol. 40, pp. 99-126, 2017.
BOOLOS, G. "The logic of provability". Cambridge University Press, 1995.
BUACAR, N. "Lógica, justificación y normatividad". *Análisis Filosófico*, Vol. 40, pp. 134-159, 2021.
BURGESS, J. P. "Which modal logic is the right one?". *Notre Dame Journal of Formal Logic*, Vol. 40, Nr. 1, pp. 81-93, 1999.
CARET, C. R. "The collapse of logical pluralism has been greatly exaggerated". *Erkenntnis*, Vol. 82, Nr. 4, pp. 739-760, 2017.
CARNIELLI, W., RODRIGUES, A. "What contradictions say (and what they say not)". *CLE e-Prints*, Vol. 12, Nr. 2, 2012.
______. "Paraconsistency and duality: between ontological and epistemological views". *The Logica Yearbook*, pp. 39-56, 2015.
______. "An epistemic approach to paraconsistency: a logic of evidence and truth". *Synthese*, Vol. 196, Nr. 9, pp. 3789-3813, 2019.
______. "Inferential semantics, paraconsistency, and preservation of evidence". In: PRIEST, G. *Graham Priest on Dialetheism and Paraconsistency*, pp. 165-187. Springer, 2019.

______. “On epistemic and ontological interpretations of intuitionistic and paraconsistent paradigms”. *Logic Journal of the IGPL*, 2019.
COBREROS, P. *et al***.** “Tolerant, classical, strict”. *Journal of Philosophical Logic*, Vol. 41, Nr. 2, pp. 347-385, 2012.
DA COSTA, N. C. A. “On the theory of inconsistent formal systems”. *Notre Dame Journal of Formal Logic*, Vol. 15, Nr. 4, pp. 497-510, 1974.
______. “Ensaio sobre os fundamentos da lógica”. Hucitec, 1980.
DEAN, W. “Montague’s paradox, informal provability, and explicit modal logic”. *Notre Dame Journal of Formal Logic*, Vol. 55, Nr. 2, pp. 157-196, 2014.
DUTILH NOVAES, C. “The undergeneration of permutation invariance as a criterion for logicality”. *Erkenntnis*, Vol. 79, pp. 81-97, 2014.
FREGE, G. “The thought: a logical inquiry”. *Mind*, Vol. 65, Nr. 259, pp. 289-311, 1956.
GLANZBERG, M. “Logical consequence and natural language”. In: *Foundations of Logical Consequence*, pp. 71-120, 2015.
GÖDEL, K. “An interpretation of the intuitionistic propositional calculus”. In: DAWSON JR., J. W.; FEFERMAN, S.; KLEENE, S. C.; MOORE, G. H.; SOLOVAY, R. M.; VAN HEIJENOORT, J. (eds.). *Kurt Gödel: Collected Works*, Vol. I, 1986.
HALBACH, V. “The substitutional analysis of logical consequence”. *Noûs*, Vol. 54, Nr. 2, pp. 431-450, 2020.
HALLDÉN, S. “The logic of nonsense”. Uppsala Universitets Årsskrift, 1949.
KENNEDY, J., VÄÄNÄNEN, J. “Squeezing arguments and strong logics”. In: *15th International Congress of Logic, Methodology and Philosophy of Science*. College Publications, 2017.
KREISEL, G. “Informal rigour and completeness proofs”. In: *Studies in Logic and the Foundations of Mathematics*, Vol. 47, pp. 138-186. Elsevier, 1967.
LO GUERCIO, N., SZMUC, D. “Remarks on the epistemic interpretation of paraconsistent logic”. *Principia: An International Journal of Epistemology*, Vol. 22, Nr. 1, 2018.
MASSOLO, A. “The normative role of logic for reasoning”. *Theoria*, Vol. 38, Nr. 2, pp. 137-154, 2023.
PINTER, C. “The logic of inherent ambiguity”. In: ARRUDA, A. I., DA COSTA, N. C. A.; SETTE, A. M. *Proceedings of the Third Brazilian Conference on Mathematical Logic*. Sociedade Brasileira de Lógica / USP, 1980.
PRIEST, G. “The logic of paradox”. *Journal of Philosophical Logic*, Vol. 8, Nr. 1, pp. 219-241, 1979.
______. “Doubt truth to be a liar”. Clarendon Press, 2005.
______. “In contradiction”. Oxford University Press, 2006.
______. “Some comments and replies”. In: *Graham Priest on Dialetheism and Paraconsistency*, pp. 575-675, 2019.
PRIEST, G., BERTO, F., WEBER, Z. “Dialetheism”. In: ZALTA, E. N., NODELMAN, U. (eds.). *The Stanford Encyclopedia of Philosophy*, Fall 2018.
RIPLEY, E. “Paradoxes and failures of cut”. *Australasian Journal of Philosophy*, Vol. 91, Nr. 1, pp. 139-164, 2013.

RUSSELL, G. "Logic isn't normative". *Inquiry*, Vol. 63, Nr. 3-4, pp. 371-388, 2020.
RODRIGUES, A., CARNIELLI, W. "On Barrio, Lo Guercio, and Szmuc on logics of evidence and truth". *Logic and Logical Philosophy*, Vol. 31, Nr. 2, pp. 313-338, 2022.
SHAPIRO, S. "Logical consequence, proof theory, and model theory". In: *The Oxford Handbook of Philosophy of Mathematics and Logic*, pp. 651-670. Oxford University Press, 2005.
SMITH, P. "Squeezing arguments". *Analysis*, Vol. 71, Nr. 1, pp. 22-30, 2011.
SOLOVAY, R. M. "Provability interpretations of modal logic". *Israel Journal of Mathematics*, Vol. 25, Nr. 3-4, pp. 287-304, 1976.
STALNAKER, R. "On logics of knowledge and belief". *Philosophical Studies*, Vol. 128, Nr. 1, pp. 169-199, 2006.
STEI, E. "Rivalry, normativity, and the collapse of logical pluralism". *Inquiry*, Vol. 63, Nr. 3-4, pp. 411-432, 2020.
TAJER, D. "Derivative normativity and logical pluralism". *Asian Journal of Philosophy*, Vol. 3, Nr. 2, pp. 1-14, 2024.
TAJER, D., FIORE, C. "Logical pluralism and interpretations of logical systems". *Logic and Logical Philosophy*, Vol. 31, Nr. 2, pp. 209-234, 2022.
VAN DALEN, D. "Intuitionistic logic". In: *The Blackwell Guide to Philosophical Logic*, pp. 224-257, 2017.
VAN DALEN, D., TROELSTRA, A. S. "Constructivism in mathematics: an introduction", Vol. I. North-Holland, 1988.