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Article

A Development of the Principle of Virtual Laws and its Conceptual Framework in Mechanics as Fundamental Relationship between Physics and Mathematics

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Abstract:

Generally speaking, *virtual displacement* or *work* concerns to a timely idea according to which a motion of a certain body is not the unique possible motion. The process of reducing this motion to a particular magnitude and concept, eventually minimizing as a *hypothesis*, can be traced back to the Aristotelian school. In the history and philosophy of science one finds various enunciations of the *Principle of Virtual Laws* and its *virtual displacement* or *work* applications, i.e., from Aristotle to Leibniz's *vis viva*, from Maupertuis' *least action* to Euler and Lagrange with calculus of variations (statics and dynamics) to Lazare Carnot's mechanics. In this case study, I will demonstrate that a particular approach used by Lazare Carnot is original by explaining within the historical context of rival approaches such as the development of the *Principle of virtual Laws* (also known as the *Principle of virtual velocity* or of *virtual work*). I will also discuss Carnot's *geometric motion* as one of the possible but invertible movements applied to *virtual displacement* as employed in his theories of machines and collisions. I will then go on to explore how the originality of an *invertible motion* within his mechanical and, in general, mathematical research program permitted Carnot to introduce a new way of structuring a scientific theory and making mechanics, with respect to the Newtonian paradigm, to scholars and his students of the *École polytechnique de Paris*.

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Keywords:

Virtual laws; virtual velocities; Aristotle; de Nemo; Tartaglia, *gravitas secundum situm*, Newtonian paradigm, Lagrange's mechanics, Geometric motion, Mechanics and collision theory, Lazare Carnot, Relationships between Physics and Mathematics, Foundations of Physics.

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Introduction

A history of the *Principle of virtual laws* (velocities, work) asserts, i.e., that even if one agrees that the *principle of virtual work* exist prior to all laws of mechanics (and not all scholars agree with this thesis), and could therefore be derived from the *principle of virtual work*, in the end, the fact that the *principle of virtual*

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work is self-evident cannot be accepted. In other words, one cannot accept it as a mere principle. Therefore, proof is necessary; or, a reduction of a theorem of another approach to mechanics, or an attempt to provide a more convincing version. Consequently, the main challenge of proving a *proof* of the *principle of virtual work* sparked a heated debate, especially with Vittorio Fossombroni (1754–1844; Fossombroni 1794) and in France where Lazare Carnot (Carnot L 1786, 1803a), Fourier (Fourier 1798; 1888–1890, 475–521), Ampère (Ampère 1806) and Poinsot (Poinsot 1838; see also Poinsot 1806) provided major contributions. In effect, a difficulty was linking the problem to Newtonian laws and obtaining its formal validity. Initially, this principle was independent from Newtonian laws, which were generally concerned with an isolated particle (or the systems derived from it). The *Principle of virtual work* also deals with extended systems of bodies² which include constraints in an essential way.³ These given forces are constraining reactions that are not included in the classical Newtonian scheme because they are unknown *a priori* (Lagrange 1788, pt II, IV). It was on the principle of least action that young Lagrange concentrated his attention.

Before beginning with the following scientific and historical overviews explanation, just few lines to clarify that the discussion will focus on: a) the history and philosophy of sciences, precisely on the historical epistemology of physics (mechanics) and its relationships with mathematics; and b) a mechanical system subject to constraints. In particular, the latter is a standard physical system⁴ today which deals with movement (kinematics), equilibrium (statics), applied physical forces (dynamics) and in some circumstances, applied mechanics, including machines, machinery and micro elements. The constraints (also *vincula*) result in a restriction on the freedom of movement of a mechanical system, i.e., of particles. The constraints are presented as mathematical physics relationships between coordinates and force. Thus, considering these aspects of a mechanical system, I am going to develop historical and epistemological research, which relies upon historical and historiographical (Kragh 1987) sources. My aim is to suggest *history and philosophy of science as historical epistemology of science* (Renn 1995; Renn and Damerow 2010).

Related to this, Lazare Carnot presented a new mechanics system without axioms and metaphysical entities. When he worked on machines, he analyzed the efficiency of machines using the *Principle of virtual laws* and reasoning about a collision model of mechanical interaction. Despite his fundamental works (Carnot L 1786, 1803a,b, 1813) and his notable political career (Dhombres and Dhombres 1997), Lazare Carnot appears as minor figure in comparison with lead scientists such as Newton, D'Alembert, Lagrange and Fourier. This is a general conviction, which deserves to be discussed. By investigating historical and epistemological aspects of science foundations, particularly focusing on the relationship between physics and mathematics, we open to the possibility of discovering new insights.

The Structure of the Paper

In order to discuss the originality of the *Principle of virtual work* adopted and used by Lazare Carnot in his mechanics (and geometry), an early historical account about the development of this principle is necessary. Therefore, the paper is structured as follows:

- Early approach: *Principle of virtual laws* by Aristotle (virtual velocities in *Problemata mechanica*⁵) and applications by de Nemore and Tartaglia (*Science of weights, virtual velocities, virtual displacements*).

²A difference exists with Euler's reasoning on fluids (Euler 1757, 286): the partial derivatives in Euler's equations (Euler's fluids) are applicable to compressible as well as to incompressible flow. It consists of an application of either an appropriate equation of state or assuming that the divergence of the flow velocity field is zero, respectively.

³ Some of this historical material is clearly dealt with Darrigol's *Physics and Necessity* (Darrigol 2014, 80-84).

⁴ Generally speaking and nowadays, a physical system is defined as the portion of the physical universe chosen for physical investigations and mathematics modeling.

⁵ Attributed to Aristotelian School. I remark, in any case, that the *equilibrium* is never cited in *Problemata mechanica*. On that a remark is only reported by the commentator (Aristotle 1955 [1936], 350, ft. a). See also Duhem 1905-1906, I, 240-270.

- Late approach: mechanics in context, principle of *virtual work* by Lagrange (*analytical mechanics*).
- A case study on the relationship between physics and mathematics: Lazare Carnot (*equilibrium and movement, machines, collision theory*).

For sake of brevity,⁶ a description of the different mechanical traditions in the seventeenth and eighteenth century are not presented in the details. Readers interested in previous works of mine can refer to: Pisano and Capecchi 2013, Gillispie and Pisano 2014, and most recently, Pisano and Capecchi 2015. However, taking in to account the objective of the paper, the extended part at the beginning on the *Principle of virtual work/displacements* and subsequent parts are largely discussed. For sake of brevity and the aims of this paper, I do not go into to the huge literature around both Bernoulli-Varignon-D'Alembert conceptualisations (parallelograms, energies, forces, velocities, theorems/laws, dynamics/mechanics as empirical science, geometry, etc.: i.e., see Bernoulli 1686, 1703, 1742; Varignon 1725; D'Alembert 1758, 1755 [1743]) and the debate on its proof belonging/not belonging to the Newtonian (paradigm) laws (i.e., see Sommerfield 1951, 1952).

A Scientific Overview of the *Principle of Virtual Laws*

In order to define the principle of virtual law in scientific terms, a short overview follows.

In modern terms, to define the *principle of virtual work*, one can specify that a *displacement* is possible if it is compatible within the fixed constraints. In addition, it is *virtual* if it is compatible within the constraints even if they are moving. Specifically, *virtual displacement* δs of a point is any arbitrary infinitesimal change in the position of the point consistent with the constraints imposed on the motion of the point. This displacement can be only imagined. Limiting ourselves to the case of time-independent constraints, we can also derive a *possible displacement* (i.e., a rotation as well). If the *virtual displacements* are mathematically independent variables, they are also arbitrary.

The *principle of virtual work* – which arises in the application of the *Principle of least action* (hereafter discussed) to the study of forces and movement of a mechanical system – is a law of mechanics for which there is not any generally accepted epistemological and ontological account: it can be seen both as a principle and as a theorem to be proven. The *principle of virtual work* is used in statics for a solution of a special class of problems involved in a system in equilibrium. Generally speaking:

The necessary and sufficient condition for equilibrium of a mechanical system without friction is that the virtual work done by the externally applied forces f is zero.

$$\sum f_i \cdot dr_i = 0$$

On the whole:

- a) The necessary and sufficient condition for the equilibrium of a particle is zero virtual work done by all working forces acting on the body during any virtual displacement δs consistent with the constraints imposed on the particle;

⁶ The *Principle of Virtual Laws*, argument of this paper, is a fundamental part of my lines of research. In ca. two last decades, I met it in several studies of mine on the relationship between physics and mathematics in history and philosophy of science such as my main subject of inquiring. This paper is an advanced-structured reorganization of a short self-sufficient interlude and spots-ideas of the *Principle of Virtual Laws* presented in previous publications of mine. In advance, I warmly thank the publishers. Generally speaking I dealt with it in *Conceptual and mathematical structures of mechanical science between 18th and 19th centuries* (Pisano and Capecchi 2013), *Lazare and Sadi Carnot: A Scientific and Filial Relationship* (Gillispie and Pisano 2014, 353-356; 375-381), *Tartaglia's science of weights* (Pisano and Capecchi 2015, 224-225), *The Emergencies of Mechanics and Thermodynamics in the Western Society during 18th-19th Century* (Pisano and Bussotti, 412-413). Thus, to deeply discuss and to present a structured history of the principle of virtual laws from Aristotle to Lazare Carnot – the cited publications – as cases study – necessary parts are cited from them as a self-citation. Particularly, I thank Springer (Dordrecht) for courtesy and permission.

- b) The necessary and sufficient condition for the equilibrium of a rigid body is that the virtual work done by all external forces acting on the particle during any virtual displacement δs consistent with the constraints imposed on the body is equal to 0.

Particularly, if the *principle of virtual work* is applied to a system of rigid bodies (i.e., mechanism), then no *virtual work* is done by internal forces, by reactions in smooth constraints, or by forces normal to the direction of motion. The virtual work is done by reactions when friction is present.

In statics, it is possible to calculate a force F implied in an equilibrium state concerning, e.g., a crank-slider mechanism in the position given by a certain known angle. For example, by knowing essentially a) the position of points of actions of applied forces implied in a given mechanical system, b) the coordinates-position (α, β, γ) of the crank (and distances and others data of the problem) then, according with application of the *Principle of virtual work*, an immediate equilibrium equation can be written using $\delta\alpha$, $\delta\beta$ and $\delta\gamma$.

Early Times

Based on two main traditions of the *principle of virtual work*, Aristotle (384-322 BC) deals with *virtual velocities* and Jordanus de Nemore (fl. 12th–13th) deals with *virtual displacements*. Particularly in Aristotle, small forces can move great weights:

[On the lever, Problem 3]. Why is it that small forces can move great weights by means of a lever, as was said at the beginning of the treatise, seeing that one naturally adds the weight of the lever? For surely the smaller weight is easier to move, and it is smaller without the lever. Is the lever the reason, being equivalent to a beam with its cord attached below, and divided into two equal parts? For the fulcrum acts as the attached cord: for both these remain stationary, and act as a centre. But since under the impulse of the same weight the greater radius from the centre moves the more rapidly, and there are three elements in the lever, the fulcrum, that is the cord or centre, and the two weights, the one which causes the movement, and the one that is moved; now the ratio of the weight moved to the weight moving it is the inverse ratio of the distances from the centre. Now the greater the [image] distance from the fulcrum, the more easily it will move. The reason has been given before that the point further from the centre describes the greater circle, so that by the use of the same force, when the motive force is farther from the lever, it will cause a greater movement.⁷

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The main Aristotelian *simplicity hypothesis*, concerning virtual work and correlated *Principle of Last Action* (see the notable Feynman's lectures on the subject: Feynman 1963), was based on the actual movement of a body as a natural motion, tending to minimize the motion of a particular material body (i.e., one can see Aristotelian inertia). In Jordanus⁸ de Nemore's mechanics (de Nemore 1565a [1533]), also known as the principle of virtual displacement, a physical system (e.g., masses subjected to forces) is in equilibrium state if and only if the (forces-)weights are inversely proportional to their virtual displacements. In his words:

If two weights descend along diversely inclined planes, then, if the inclinations are directly proportional to the weights, they will be of equal force when descending [idem force – equilibrium]⁹.

⁷ *Problemata Mechanica* 850a 30 in Aristotle 1955 [1936], *Mechanical Problems*, 3, 353. Renn, Damerow and McLaughlin 2003.

⁸ See also Jordanus de Nemore in Tartaglia's edition (Tartaglia 1565b; see also Clagett and Moody). Moreover, both *Elementa Jordani super demonstrationem ponderum* (1229) and *Liber de ratione ponderis* (fl XIII c.) show an interesting proof of Archimedes' law of the lever, *Quaestio Sexta* (*Liber de ratione ponderis*, in Tartaglia's *Jordani Opvscvlvm de Ponderositate* edition: Tartaglia 1565b, 5(–6), line 13), by means of the application of the *principle of virtual work* and where the fall of geometric directions of *displacements* is considered vertical by Jordanus de Nemore. Moreover, one can also see Jordanus de Nemore's *Suppositio Sexta* (Tartaglia 1565b, 1, line 13; see also *Liber de ratione ponderis* edited by Clagett and Moody, 174–175) where one can read that a body is able to raise another lighter body if a lever is utilized, that is, like an embryonic engine.

⁹ "Quaestio decima. Si per diversarum obliquitatum via duo pondera descendant, fueritque declinationum et ponderum una proportio eodem ordine sumpta, una erit utriusque in descendendo [idem force – equilibrium]" (Tartaglia 1565, 7, line 1). See also Tartaglia 1554, Quesito XV–Def. XII; Quesito XVI–Def. XIII, 84, line 7 (Pisano and Capecchi 2008, 2010; Capecchi and Pisano 2010a, 2010b; Pisano 2009c, 2008).

Particularly Jordanus de Nemore¹⁰ studied *principles of virtual displacement* in the following way (Tartaglia 1565b, *Quaestio Sexta*, pp 5–6; see Fig. 1):

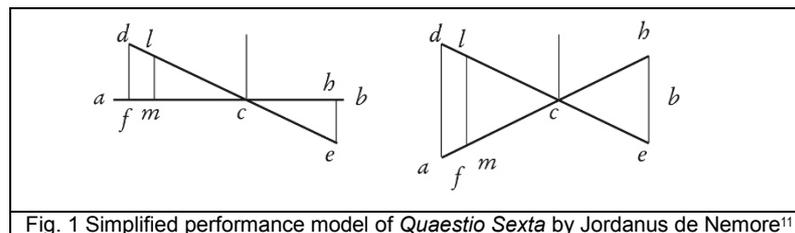


Fig. 1 Simplified performance model of *Quaestio Sexta* by Jordanus de Nemore¹¹

The *principles of virtual displacement*, adapted to force(–weight) and lever, claim that a (virtual) rise h of a body p placed on an arm of the lever should correspond to a (virtual) lowering H of a body P placed on the other arm of the lever, so that the relation $ph=PH$ should be valid. In short, two facts related to Jordanus de Nemore’s statement (*Ibidem*) should be noted:

- a) No reference to the time during *virtual displacement* is considered.
- b) De Nemore’s principle suggests two trends. If the body goes up, instead going down, then the *virtual work* is negative.

Therefore, the principle proposed by Jordanus de Nemore is a principle of equivalence, or conservation, which is not in a condition of equilibrium. In other words, the equilibrium condition must be obtained for *reductio ad absurdum* (as below discussed). The examination of the proof of the law of the lever reported by de Nemore, in his *Liber de ratione ponderis* (*Jordani Opusculum de Ponderositate* edition: Tartaglia 1565b, pp 5–6), addresses this elegant method in order to improve the theory and make it more profound. Nevertheless, we should note that the *principle of virtual displacement* is not mentioned explicitly; it is part of a proof. For instance, the proof of an inclined plane obtained by implementing the *principle of virtual displacement* is an emblematic reasoning and thanks to that principle, the procedure is correctly analysed for the first time. For an application of *Principle of virtual work* within *Science of weights* and related to the lever we have: let us consider a lever where two bodies having mass P and Q are applied at the end points of said lever. These bodies are inversely proportional to the length of the arms.

This statement is commonly expressed by $M=Fd$, where F is, e.g., one of the two force–weights ($F=mg$, related to one of the masses), d is the distance between the two force–weights and the fulcrum, and M is the consequent turning force¹². Based on the principle of equivalence, mass P can be substituted by a body having mass equal to Q , at distance q from the fulcrum of the lever, and placed on the same side of P , since $Qq=Pp$ should be valid. In this way, thanks to the *principle of sufficient reason*¹³. Nevertheless, this situation logically implies that the lever was also in an equilibrium state before substituting mass P with mass Q . In order to better introduce the *principle of virtual work*, let us examine an application of Jordanus de

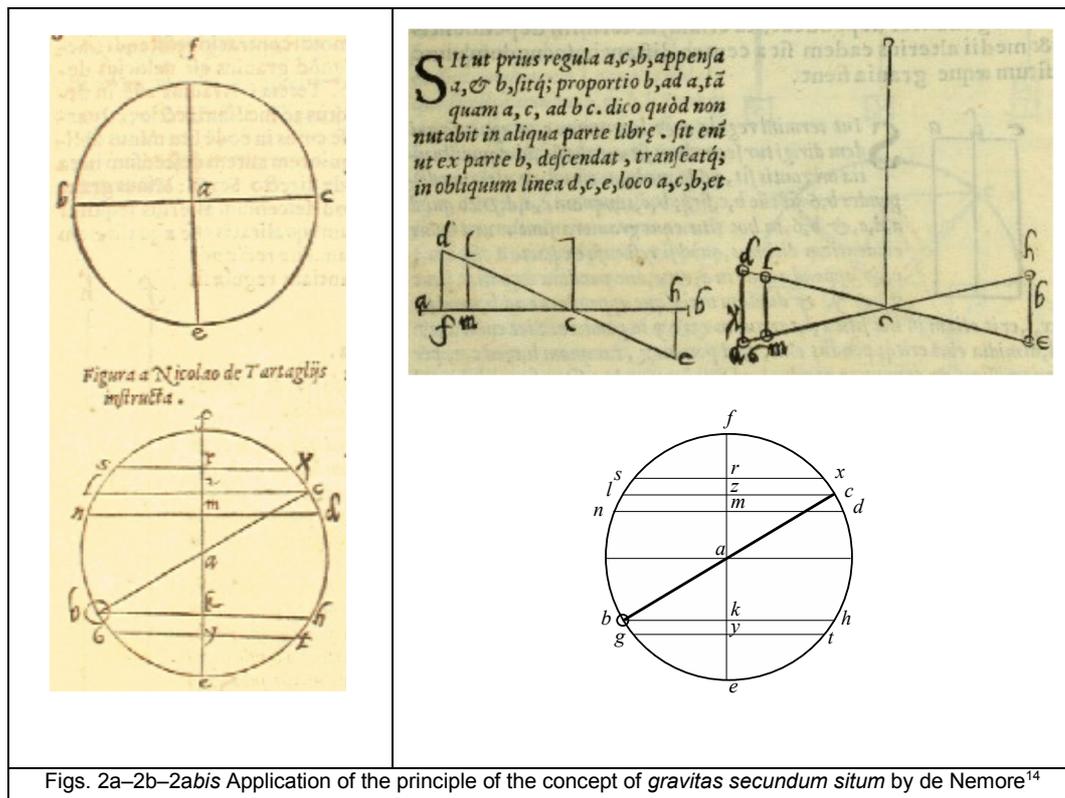
¹⁰ Jordanus de Nemore’s ideas will have a foundation in René Descartes’ (1596-1650) algebra.

¹¹ Tartaglia 1565b, 5–6. See also Pisano and Capecchi 2015, 224.

¹² I precise that M is the moment (or rotation). It is not the *Momentum*. The moment generally corresponds to a measure of an effect caused by a physical quantity around a certain axis. The *Momentum* (Galileo 1890–1909, II, 159–161) is a physical property. Nowadays, in classical mechanics is the product of the mass and velocity of an object: $p=mv$. On *Momentum* see also Galluzzi (Galluzzi 1970) and Galluzzi and Torrini (Galluzzi and Torrini 1975–1984).

¹³ Where, i.e., anything that happens does so for a reason no state of affairs can be obtained, and no statement can be true unless there is sufficient reason why it should not be otherwise. This principle is implicitly used until the born of the modern statics. For sake of brevity I do not discuss the evident importance of this principle reminding to the reader to so extensive secondary literature. I only precise that the bases of the first Archimedes’ law (*Equilibrium Plane*, Archimedes 2002) are mainly two: 1) epistemological one related with two equal weights et equal distances – that is *principle of sufficient reason* – and 2) the two weights are in equilibrium (Capecchi and Pisano 2010a).

Nemore's reasoning (Pisano and Capecchi 2007; Capecchi and Pisano 2007, 2008, Pisano and Capecchi 2013):



Let us now consider de Nemore (and Tartaglia's) application to the principle to *gravitas secundum situm* (positional gravity). Let us assume that the body having mass P moves down along the arc of the circumference (Fig. 2c) from point h to point z , thereby describing the arc of circumference hz . The path traced by P , not along the arc of circumference, but along the diameter of the circle (Fig. 2c), is $h'z'$. Additionally, let us assume, e.g. that the body having mass P , in another circumstance moves down along the arc of circumference $ak=hz$ (Fig. 2c) where the path traced along the diameter of the circle is ok' . Therefore, since $hz=ak$, we obtain that

$$h'z' < ok'$$

so the fall of the body having mass P along the diameter (Fig. 2c) is more "oblique" because it travels over a shorter segment ("less direct path"). Of course, *oblique*, means that body has more/less *gravitas secundum situm* with respect to another body-position along the circumference; it depends on the geometry of the problem. Here, for my aim related to the Carnot case study, it is no longer necessary to deal with it¹⁵. Just to mention "Ingenious reasoning, but wrong" like Marshall Clagett (1916–2005) claimed in his *The science of mechanics in the middle ages* (Clagett 1959, 76, line 18). In other words, Jordanus de Nemore – in *Elementa Jordani super demonstrationem ponderum* and partially also in *De ratione ponderis* – improperly applied the concept of *positional gravity (gravitas secundum situm)* when he reasoned upon displacements along a circumference (Capecchi and Pisano 2008; Pisano 2011).

¹⁴ As reported by Tartaglia in *Jordani opuscolvm ponderositate* (published *posthumous*, 1565): Tartaglia 1565b, pp. 3–5, p. 7. The figures should be read from left to right. *2abis* is a simplified performance model of concept of *gravitas secundum situm* by Jordanus de Nemore. (Source: Public domain, Biblioteca Viganò, Brescia, Italy)

¹⁵ For an extensive account see Pisano and Capecchi 2015.

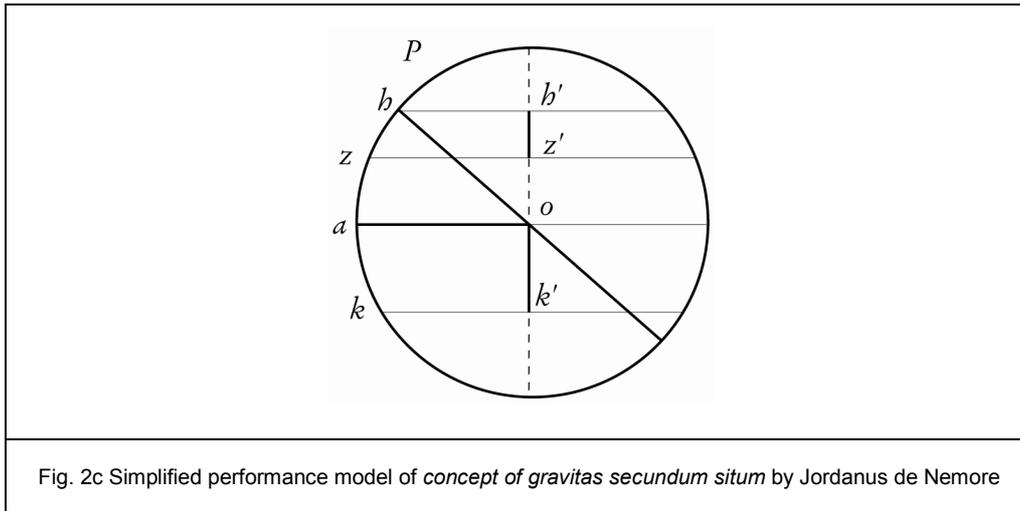


Fig. 2c Simplified performance model of concept of gravitas secundum situm by Jordanus de Nemore

The reasonings (Pisano 2007) exposed by Jordanus de Nemore in both *Liber de ratione ponderis* and in *Iordani Opvscvlvm de Ponderositate* (Tartaglia 1565b, 3–5, see also 7), and by Tartaglia in *Quesiti ed inventioni diverse* (Tartaglia 1554, pp 89–93) are very interesting. Let us look at some examples.

In the *Suppositio V*¹⁶ in *Iordani Opvscvlvm de Ponderositate* edited by Tartaglia (and in *Liber de ratione ponderis* as well) Jordanus (thus Tartaglia) reasons that a rectilinear segment intercepted along with the vertical component of the arc's virtual path suggests the *principle of virtual displacement*. Nevertheless, it is easy to verify that this reasoning is only valid if vertical segments are used. If one considers the entire diameter as the lever's beam and considers small arcs (ca. infinitesimal) along the circumference, then the difference of their vertical projections (at limit values) is zero. Therefore, the displacements become the circumference and mutually them. Therefore, vertical components are also equal. Let us examine some details of Tartaglia's (or de Nemore's) reasoning in *Quaestio sexta of Iordani Opvscvlvm de Ponderositate* (Tartaglia 1565b, 5) and in *Propositione V and VI of Quesiti ed inventioni diverse* (Tartaglia 1554, XXXII–XXXIII, Props V–VI, pp 89–91, XLI, Prop. XIII, 96–97). Tartaglia edited Jordanus de Nemore's reasoning on the equilibrium of a lever using principles of virtual work. Let there be a lever ACB (Fig. 3) having fulcrum C and, at the end of it, A and B are applied to two bodies a and b (Fig. 3). The following relation is therefore valid:

$$b : a = AC : BC \quad (1)$$

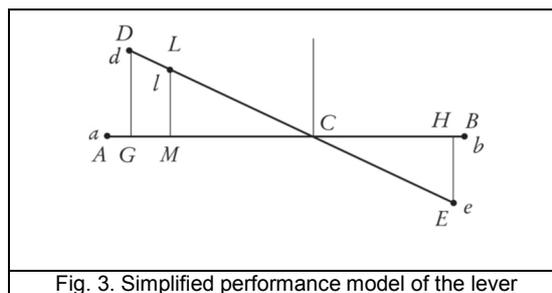


Fig. 3. Simplified performance model of the lever

Therefore, the lever is in an equilibrium state. From an epistemological point of view, with regard to Fig. 3, an *ad absurdum* proof is based on the following two main assumptions:

- 1) Based on common experience, in a physical lever system the masses (or force-weights) are inversely proportional to the length of the arms. Therefore, it assumes the following configuration: (a) equilibrium or (b) non-equilibrium. For situation (b) a necessary and logical

¹⁶ “Quinta: Obliquiorem autem descensum, in eadem quantitate minus capere de directo [A more oblique descent is one which, in the same distance, partakes less of the vertical]”. (Tartaglia 1565b, 3, line 12). See also *quarta suppositio*, “Quarta: Secundum situm grauius esse cuius in eodem situ minus obliquus descensus” [It is heavier in position when in that position its path of descent is less oblique]. (Tartaglia 1565b, 3, line 10).

condition follows: one of the two masses at the end of the lever produces an inclination of the lever toward its side.

- 2) If either of the masses descends, then it raises a mass equal to itself which is distant from the fulcrum, creating a displacement equal to that of its descent.

Let us describe, e.g., by *reductio ad absurdum*, equation (1) assumed in the above-cited configuration (b) at point 1): the lever is not in an equilibrium state. This is the *ad absurdum* hypothesis.

Therefore, in that condition of non-equilibrium, the lever should necessarily incline toward one of two sides. Let us suppose that it inclines toward *B* (Fig. 3). In this way, the physical system lever-masses gets the new configuration *DCE* (Fig. 3). If we consider the same masses of previous configuration *ACB*, ($d=a$ in *D* and $e=b$ in *E*), then it is possible to draw the perpendicular *DG* on *AC* in descent. Therefore, by the same method of drawing, the rise of the perpendicular *EH* on *CB* is possible. It is clear that both the descent and rise along the arcs of circumference

$$AB \quad \text{and} \quad DE ,$$

and rectilinear displacements

$$\overline{AB} \quad \text{and} \quad \overline{DE}$$

are considered¹⁷. In Fig. 3, due to the law of the similitude of triangles *DCG* and *ECH*, one can obtain:

$$DC : CE = DG : EH \quad (2)$$

Moreover, the following relation should also be valid:

$$DC : CE = b : a \quad (3)$$

Therefore

$$DG : EH = b : a \quad (4)$$

At this point, the author considers a point *L* on segment *DC*. The proof assumes an *ad hoc* procedure: point *L* is made to be symmetric to *E* with respect to fulcrum *C*. In *L*, a mass *l* equal to mass *b* is added. Here, it seems that Jordanus de Nemore would like to verify what happens to an imagined (virtual) motion of a certain mass in a certain point of beam during the previous configuration. In this sense, rise *e* until *B* corresponds to descent *l* until *M*. In other words, thanks to the geometric construction (and an *ad hoc* procedure) of the problem, one can obtain that $LM=HE$. Again, the law of the similitude of triangles establishes the following relation:

$$\frac{LM}{DG} = \frac{CL}{CD}$$

Therefore

$$DG : LM = b : a .$$

Since $l=b$, the following conclusion is obtained:

$$a : l = LM : DG$$

Therefore, masses *a* and *l* are inversely proportional to their vertical (opposite) displacements toward the elevated side. In effect, these displacements concern the reasoning upon the virtual rotation of the lever. At

¹⁷ Let us note that these segments are physically the vertical components of the displacement vector.

this point, Jordanus de Nemore¹⁸ reasons “Therefore, what suffices to lift a to D , will suffice to lift [mass] l through the distance LM ”¹⁹. Since $l=b$, and, also $LC=CE=CB$ equilibrium is obtained. In this reasoning, since b should not be (“sufficient”– ly) able to displace l of a quantity LM which is equal to the symmetric of b , then b , due to a logical consequence, should not even be (“sufficient”– ly) able to displace a of a quantity DG . Nevertheless, for (Eq. 1) and since it is not possible to presuppose it *a priori*, the lever cannot assume configuration DCE . It is evident that the proof is based on an Aristotelian approach, even though certain aspects differ. In fact, with respect to the Aristotelian approach, 1) which considers the displacements along circular arcs, de Nemore also reasons on *virtual* rectilinear paths; 2) in Supposition IV – “It is heavier in position when in that position its path of descent is less oblique”²⁰ – he considers the descent along an inclined plane. Thus, according to Giovanni Vailati²¹ (Vailati 1896–1897, 15; see also 1–25) Jordanus de Nemore deals with, albeit in an embryonic stage, the problem of the inclined plane (afterwards Tartaglia corrected some aspects). 3) The third instance follows. Initially, it seems that an *ad absurdum* proof was not necessary –it could have been adopted only for the sake of simplicity. That is to say, *virtual displacement* could be considered a difference of positions, or the mathematical displacements of a and of b and not as an effect of certain forces (at the time, the concept of force was not well physically and mathematically defined). In order to do so, a more explicit reasoning²² than Eq. (4) on the virtual displacement is necessary. By using the aforementioned quote: “Therefore, what suffices to lift a to D , will suffice to lift [mass] l through the distance LM (Tartaglia 1565b, 6, line 3). In effect, Eq. (4) effectively establishes a criterion of equivalence applied to the lever. In other words, the work by a certain Force–weights for a body a

$$F_p^a$$

and considering a certain displacement ($DG=rise$), is equal to work accomplished by the same Force–weights for a body b

$$F_p^b$$

to displace the same quantity ($LM=rise$), the body b symmetrically positioned to a with respect to the fulcrum. From a mathematical point of view (Fig. 1) and by considering virtual displacement extended to the principle of virtual work, one can obtain:

$$L^a = (F_p^a \cdot DG) = (F_p^b \cdot LM) \tag{5}$$

The two works, thanks to different orientations of the displacements, have opposite signs, and are equal if they are considered in absolute value. Therefore, the total work is null. In this sense, the third cited proof is in conflict with Aristotle’s claim that there is a reason to claim the equality in the Equation (5). In fact, the total work is null²³. Therefore, equilibrium is assumed. It should be noted that Tartaglia’s reasoning (Tartaglia 1554, 89–91) upon *gravitas secundum situm* in *Quesiti et inventioni diverse* is similar to Jordanus de Nemore’s reasonings in *Iordani Opvscvlvm de Ponderositate*.

¹⁸ Let us remark that Tartaglia (Tartaglia 1554, Q. XXXII, Prop. V, 90–91) cited the non–exactness of the Aristotelian rule but he does not cite the previous correction made by Jordanus de Nemore.

¹⁹ “Quod ergo sufficit attollere a in D , sufficit attollere l secundum LM .” (Tartaglia 1565b, 6, line 3). In other words: “to displace a mass p to height h is equivalent to displacing a mass $q=p/k$ to height hk , whatever is k ”. This statement is, in some way, connected with the principle of the impossibility of perpetual motion.

²⁰ “Quarta: Secundum situm gravius esse quando in eodem situ minus obliquus est descensus” (Tartaglia 1565b, 3, line 10).

²¹ In this regard, Jordanus de Nemore – using some variants on the positional gravity – also applied (*Liber de ratione ponderis*) this concept to the angular lever, reasoning upon the equilibrium of an inclined plane.

²² This reasoning was criticized by Simon Stevin. The latter was opposed to it, since it was absurd to reason on the fact that a *situation of equilibrium* could be derived from a *situation of motion* (Capecchi and Pisano 2007, 2010; Pisano 2010b; Radelet de Grave 2007, 1996).

²³ On the development of engines, recently: Pisano and Bussotti 2015b.

Late Times

Explicit comments regarding the principle of virtual work are also reported by Galileo in *Le Mecaniche* (Galileo 1890–1909, II, 155–191) and in *Discorsi intorno alle cose che stanno in sù l'acqua* (Galileo 1890–1909, IV, 3–141). Particularly in this latter manuscript, Galileo clearly attributed the law of virtual velocity to Aristotle (Aristotle 1955 [1936], 847a 10–15, 847b 10, 329–332) also adding that the idea of the principle of virtual work was born thanks to the observation of the motion of points, which rotate along a circumference. Galileo also dealt with the law of virtual displacement in more than one situation²⁴.

Recent notable historical studies consider Isaac Newtonian's contributions (as a paradigm in history), as expounded in his 1687 masterpiece *Philosophiae naturalis principia mathematica* (Newton 1803 [1686–7]; Pisano and Bussotti 2016a,b), as the climax of classical science and mechanical scholars of the Enlightenment added little to it. The tendency today is to realize that this analysis is misleading and that this period, far from being a dark century, was filled with fundamental contributions which helped to establish the majority of mechanical concepts. On the other hand, the change in the role of practical mathematics in the understanding of the world (Henry 2011, 193–196; Westman 1980, Jardine 1988; Høyrup 1994) is also discussed (Capecchi and Pisano, 2010a, 2010b, 2007, 2008; Pisano and Capecchi 2013).

It is known that a great amount of Newtonian mechanics – under certain physical standpoints – only allowed the study of the motion of material points free in space with an incompletely developed mathematical apparatus which was based on calculus and methods by several scholars at that time, too. That is one of the main reasons for the emergence of the next commented editions (Bussotti and Pisano 2014a, 2015a). In fact:

The mathematization of many problems posed prematurely by Newton in the *Principia* became possible (e.g., in the moon theories of Euler, Clairault and d'Alembert developed in the 1750s) thanks to calculus which was neither Leibnizian nor Newtonian: it would be better to call it “Eulerian calculus.”²⁵

Furthermore, problems related to systems of constrained points remained unapproachable, as did the study of continuum bodies both rigid and deformable. Moreover, Newtonian science had to face, mainly on the Continent, people who were reluctant to follow his reasoning; Cartesianism was still dominant and the religious metaphysics behind his work were not well respected (Bussotti and Pisano 2013; Panza and Malet 2006). This situation promoted a profound innovation in dynamics as formulated by Newton. Newton's main concepts, that of force, remained dominant among scientists but their interpretation changed and a different approach based on work and energy became a serious contender. Competition was not based on a more or less appealing ontology but on a simpler or more complex mathematical formulation. Before beginning any considerations regarding XVIII century mechanics, it is essential to refer to Newton's own mechanics, which should be ideally located in the XVII century. Only in this way will it be possible to identify the distinctive features of Lagrange's account (Bussotti 2003) Euler, Carnot's (Gillispie and Pisano 2014) mathematics (Alvarez and Dhombres) and mechanics (Capecchi 2011, 2012). It will become clear that in what is referred today as Newtonian mechanics, only a few aspects can be recognised as Newton's proper mechanics, as defined in his *Philosophiae naturalis principia mathematica*. Newton assumed the following laws of mechanics which he referred to as laws, probably to stress that he considered them to be of an experimental nature (Newton 1803 [1686–7]):

DEFINITION III. *The vis insita, or innate force of matter is a power of resisting, by which every body, as much as in it lies, endeavours to preserve in its present state, whether it be of rest, or of moving uniformly forward in a right line.*²⁶

Axioms; or Laws of Motion. Law I. Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon; *Law II:* The

²⁴ Galileo 1890–1909 II, 240–242, IV, 68–69; VIII, 310–331, 329–330. See also: Pisano 2016, 2010a, 2009a,b,c; Pisano and Bussotti 2012, 2015a,b,c.

²⁵ Guicciardini 1999, 259. Author's italic.

²⁶ Newton 1803 [1686–7], I, 2; author's *italic* style and Capital letters.

alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed; *Law III*: To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.²⁷

These laws are quite familiar to a modern reader even though some particularities²⁸, both formal and substantial, do not go unnoticed, mainly for Law II. First, the famous formula $f = ma$, commonly known as Newton's second law, is differently proposed – with respect to today's formula. Mass is not named explicitly but is absorbed in the quantity of motion and in the end no reference is made to acceleration. Scrutiny shows that impressed force, apparently, the only element identifiable in the second law, cannot be identified with the modern concept of force. Indeed, integration of the law of motion, considered in the modern mathematical sense as $f = ma$, over a finite interval of time produces:

$$m\Delta v = \int f dt$$

where the first part is the variation of the quantity of motion, or according to Newtonian terminology, the alteration of motion. A comparison of the analytical expression just obtained with Law II of motion shows that what Newton calls force must be equal to

$$\int f dt$$

Newton chose the word *force* to indicate a founding quantity of dynamics, but he did not connect it to any of the concepts known as *force* today. Newton's concept of force is quite far from the concepts force introduced by previous scientists such as Descartes (Schuster 2000, 2013a, 2013b; Bussotti and Pisano 2013) and Torricelli (Capecchi and Pisano 2007; Pisano 2009b). This concept today is scarcely used and is not referred to by this name; the most common name for

$$\int f dt$$

that is the impulse of force f .

In *Scholium*, which followed the three laws of motion, Newton wrote about the force of gravity as an example of a force acting continuously. Stated *verbatim*:

When a body is falling, the uniform force of its gravity acting equally, impresses, in equally particle of time, equal forces upon that body, and therefore generates equal velocity; and in the whole time impresses a whole velocity proportional to the time.²⁹

That is, the total variation of velocity is proportional to the total force, which is proportional to time. In *Philosophiae naturalis principia mathematica* the whole force can also represent the intensity of an impulse and the action of continuum force, like gravity, for instance, usually described as a sequence of impulses, divided by a constant time step Δt , which approaches zero (as the sequence of impulses) goes to infinity. *Nevertheless, what kind of foundational and logical problems still remained unsolved?* Leaders of classical mechanics in the Enlightenment age, if we do not consider masters Newton and Leibniz, who were more part of the XVIII century, were Jean d'Alembert (1717–1783), Joseph Louis Lagrange (1736–1813), Leonhard Euler (1707–1783), Lazare Carnot (1753–1823), Pierre–Louis Moreau Maurice de Maupertuis (1698–1759), Jakob Bernoulli (1655–1705), Johann Bernoulli (1667–1748), Daniel Bernoulli (1700–1782),

²⁷ Newton 1803 [1686–7], I, 19-20; *Italic style* and capital letters belong to the author.

²⁸ For sake of brevity I remind discussion on Newtonian studies to my recent papers (Pisano and Bussotti 2016a,b; Pisano 2014, 2013a,b; Bussotti and Pisano 2013, 2014a,b). Particularly, I am working with (Paolo Bussotti) to a huge editorial project about rethinking the whole *Newtonian Geneva Edition* (4 vols.): critical translations from Latin to English and related commentaries for the Oxford University Press.

²⁹ Newton 1803 [1686–7], I, 22.

and Jakob Hermann (1678–1733). Each of them provided a contribution which marked the development of mechanical science. There are different, not completely distinct, points of view regarding the historical development of mechanics: the kind of mathematical approach, the kind and logical state of principles, the model of force. In the following section I will briefly present these different points of view.

At the beginning of the XVIII century Newton was surely seen as one of the most prominent mathematicians and physicists but not as the one who established dynamics in its final form. But it is known that Newton's mechanics were considered unsatisfactory by many scholars from both epistemological and ontological points of view, mostly because of his introduction of forces acting at a distance, which were considered occult entities. More fundamentally, for scientists, Newtonian mechanics was considered limited essentially to the motion of material points free in space, unsuitable for solving problems raised by the technology of the time. As an example of the opinions of the period, below are some comments by Daniel Bernoulli and Euler:

Theories for the oscillations of solid bodies that up to now authors furnished presuppose that into the bodies the single point position remains unchanged, so that they are moved by the same angular motion. But bodies suspended at flexible threads call for another theory. Nor it seems that to this purpose the principles commonly used in mechanics are sufficient, because clearly the mutual dispositions of points are continuously changing.³⁰

But as with all writings composed without analysis, and that mainly falls to be the lot of Mechanics, for the reader to be convinced of the very truth of these propositions offered, an examination of these propositions cannot be followed with sufficient clarity and distinction: thus as the same questions, if changed a little, cannot be resolved from what is given, unless one enquires using analysis, and these same propositions are explained by the analytical method. Thus, I always have the same trouble, when I might chance to glance through Newton's *Principia* or Hermann's *Phoronomiam*, that comes about in using these, that whenever the solutions of problems seem to be sufficiently well understood by me, that yet by making only a small change, I might not be able to solve the new problem using this method.³¹

Therefore, Newtonian laws alone were not sufficient for understanding all laws of motion as more fundamental mechanical laws were necessary. Generally speaking, from an epistemological point of view, the problems faced by XVIII century scientists were less demanding than those faced by Newton concerning the search for general laws, but this does not mean that they were any simpler. They concerned, for example, the search for the oscillation centre of a rigid body and the study of the vibrations of a chain or a thread. The search for the centre of oscillation was quite a relevant and difficult problem. It was equivalent to finding the length of a simple pendulum within the same period. The problem was substantially solved by Christian Huygens (1629–1695) in his *Horologium Oscillatorium sive De motu pendulorum ad horologia aptato demonstrationes geometricae* (Huygens 1673) by means of the first formulation of the theorem of living forces (*vis viva*). Jakob Bernoulli revisited the subject in 1713, with a completely different and promising approach in his paper *Démonstration générale du centre de balancement et d'oscillation, tirée de la nature du levier* (Bernoulli Jakob 1703). His study addresses the roots of both D'Alembert's principle and the angular momentum equation. Johann Bernoulli also faced these problems beginning at the end of the XVII century and published his considerations in *Opera Omnia* (Bernoulli Johann 1742a; 1742b; see also Radelet de-Grave 2009) where he first introduced the concept of angular acceleration, a very relevant idea. Scientists such as Euler, D'Alembert Johann and Daniel Bernoulli studied the problem of the vibrating chain. Among the problems that busied the minds of XVIII century scientists was the study of the motion of bodies on mobile surfaces, such as the motion of a heavy body upon an inclined plane which moves on a plane without friction. Johann Bernoulli (Bernoulli Johann 1742b) studied the motion of a material point from a Newtonian approach by introducing constraint reactions within external forces, referring to them as immaterial forces, since they were outside the bodies in contact. It should be noted that the assimilation of constraint reactions to ordinary forces was quite common in statics, but in dynamics the problem was much more complex conceptually because reactions were endowed with activity. Euler, who developed principles of mechanics which facilitated the introduction of constraint reactions, tried to avoid their explicit use as

³⁰ Bernoulli D 1733, 108.

³¹ Euler 1736, *Praefatio*, (Author's *italic* style and Capital letters). See also: Euler 1774 [1773]; 1749; 1752 [1750].

much as possible. To conclude, in the solution of the various problems, a unique principle was not referred to and it was sought in analogies with problems which had already been solved.

In the mid-eighteenth century, thanks to the brilliant minds of Euler and Johann Bernoulli, some general principles confirmed the living forces theorem and the minimum action principle. On the living forces theorem after Huygens and Leibniz, were works by Johann and Daniel Bernoulli, D'Alembert and notably Lagrange.³² The living forces theorem was only correct for problems limited to one degree of freedom, because it gave only a scalar equation. More interesting but equally incomplete was the minimum action principle. It can be attributed to Pierre de Fermat (1601–1665) who in 1662, in his studies on the refraction of light, wrote a letter addressed to de La Chambre (“Sunday, January 1, 1662”) and drew on a theological and moral principle, namely, “[...] nature *always acts along the shortest paths* [...]”³³ (Fermat 1891–1922, vol II, CXII [D., III, 5r.] 458). However, Maupertuis gave it the name and extended it to mechanics in various steps. The final step was referred to in his paper *Les lois du mouvement et du repos déduits d'un principe métaphysique* of 1746:

Because of so many people who worked about I hardly dare to say I have discovered the principle on which all these laws are based: which extends also to elastic Bodies, from which the movement of all physical corporeal depends.³⁴

That is the principle of *least quantity of action* so wise, so worthy of the Supreme Being, et to which Nature looks so constantly joined, that it observes it not only in all its changes, but also in its permanence. *In the impact of Bodies, the Motion distributes so that the quantity of action, which derives from the change, is the least possible.*³⁵

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Euler tried to give the least action principle a more precise formulation, however did not reach a general formulation, leaving to metaphysics the decision of whether or not it could be assumed as a general principle or if it could be determined within the laws of mechanics. It was Lagrange in *Application de la méthode exposée dans le mémoire précédent à la solution des différents problèmes de dynamique* (Lagrange 1870–1873 [1762], I, 363–468) who provided a proof of the least action principle based only upon the laws of mechanics, without any metaphysical traces. Despite some important successes in the solution of various problems and the existence of some general principles, a general feeling of disappointment prevailed among the scientists of this period which gave rise to an effort to find simpler and more general principles. This effort began to be fruitful in the second half of the XVIII century and was led by Euler and Lagrange to a nearly finished form, respectively, of *vectorial* and *analytical* mechanics.

Lagrange's results on his studies of the least action principle were published in *Application de la méthode exposée dans le mémoire précédent à la solution des différents problèmes de dynamique*, and represented the perfecting of its formulation and his convincing proof. The use of the principle, however, is possible only for what should be referred to today as conservative systems. In such a case, by the addition of the living force theorem, any (discrete) dynamical problem could be solved. However, even before *Application de la méthode exposée dans le mémoire précédente à la solution des différents problèmes de dynamique*, Lagrange thought of a principle which was more general than the least action principle. In a letter to Euler on November 24th 1759, Lagrange³⁶ wrote about having composed elements of differential calculus and mechanics to develop the true metaphysics of his principle. In *Recherches sur la libration de*

³² Johann Bernoulli, *Theoremata selecta pro conservatione virium vivarum demonstranda et experientia confirmanda* (Bernoulli Johann 1727), Daniel Bernoulli, *Remarques sur le principe de la conservation des forces vives pris dans un sens général* (Bernoulli D 1750 [1748]). D'Alembert's *Traité de dynamique* (D'Alembert 1758 [1743]; see also Hankins 1970; D'Alembert 1751–1780; 1767) and Lagrange's *Recherches sur la libration de la Lune* (Lagrange 1764). On that and a development of mechanics between 17th and 18th century is previously published (Pisano and Capecchi 2013, 97–121). On the history of mechanics (Truesdell 1968a,b) and also history of thermodynamics (Truesdell 1970, 1976, 1980) Truesdell's works (sometime considered by someone *surpassed*) are – in any case – important examples of historical criticism and reflections.

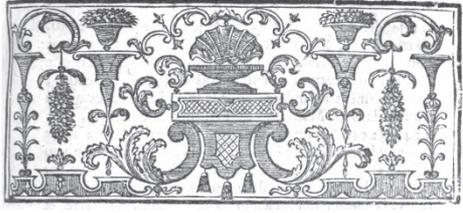
³³ Author's *italic* style.

³⁴ Maupertuis 1746, 286. (Author's *italics* style).

³⁵ *Ibidem*.

³⁶ Euler's Correspondence with Joseph Louis de Lagrange. In: *Opera Omnia*. Series IV, Vol I, retrieved via: <http://eulerarchive.maa.org/correspondence/correspondents/Lagrange.html> *Idem* letter was previously edited in Lagrange 1892, Tome XIV, pp 170–174.

la Lune (Lagrange 1764), for the first time, Lagrange obtained dynamical equations of motion with the aid of a new law of Mechanics, the principle of virtual work:

<p style="text-align: center;">1764</p>  <p style="text-align: center;">RECHERCHES SUR LA LIBRATION DE LA LUNE, Dans lesquelles on tâche de résoudre la Question proposée par l'Académie Royale des Sciences, pour le Prix de l'année 1764.</p> <p style="text-align: center;">I.</p> <p>ET écrit a pour objet d'examiner les différens mouvements apparens, ou réels que la Lune peut avoir autour de son centre. Je suppose d'abord que cette Planette à une figure quel- conque; & je cherche le mouvement qu'elle doit recevoir de l'action de la terre & du soleil. Quoi qu'un très- grand Geomètre ait déjà donné des méthodes & des formules générales, qui peuvent aisément s'appliquer à la recherche dont il s'agit ici, néanmoins il m'a paru plus commode de reprendre la question en entier, & de</p> <p style="text-align: right;">A</p> <p style="text-align: right;">Prix de l'Acad. Tome LX.</p>	<p>III.</p> <p>“There is a principle generally true in Statics, according to which, if a system of whichever bodies or points, each of them subjected to powers [forces], is in equilibrium and if somebody gives the system a small arbitrary motion, for which each point covers an infinitesimal space, the sum of the powers multiplied each of them for the space covered by the point to which it is applied, in the direction of this power, will be always = 0 [zero].³⁷</p> <p>[In the same article “III” he used the following formulas:]</p> <p>Dans la question présente, si on imagine que les lignes X, Y, Z, R, R', deviennent, en variant infiniment peu la position de la Lune autour de son centre, $X + \delta X, Y + \delta Y, Z + \delta Z, R + \delta R, R' + \delta R'$, il est facile de voir que les différences $\delta X, \delta Y, \delta Z, \delta R, \delta R'$, exprimeront les espaces parcourus en même tems par le point a dans des directions opposées à celles des puissances $\alpha \frac{d^2 X}{dt^2}, \alpha \frac{d^2 Y}{dt^2}, \alpha \frac{d^2 Z}{dt^2}, \alpha \frac{T}{R^2}, \alpha \frac{S}{R'^2}$, qui sont sensées agir sur ce point; on aura donc, pour les conditions de l'équilibre, l'équation générale.</p> $\int \left(\alpha \frac{d^2 X}{dt^2} X - \delta X + \alpha \frac{d^2 Y}{dt^2} Y - \delta Y + \alpha \frac{d^2 Z}{dt^2} Z - \delta Z + \alpha \frac{T}{R^2} X - \delta R + \alpha \frac{S}{R'^2} X - \delta R' \right) = 0.$ $\left[\sum f_i \cdot \delta u_i = 0 \right]$
<p style="text-align: center;">Fig. 4 Frontispiece of the <i>Recherches sur la libration de la Lune</i> and his formulation³⁸</p>	

The progress of the Lagrangian formulation (Pisano 2014, 2013a) of the principle over that of Johann Bernoulli in 1715 was significant in many ways. Lagrange’s formulation was stated in a clearer way for a system of bodies, because only virtual displacements congruent with constraints were used, and also because the principle was embedded in the newly established *variational* calculus. At any rate, the principle

³⁷ Lagrange 1764, 5 (Author’s Capital letters). See also Lagrange’s arguments on the same topic in the *Théorie de la libration de la lune*: “[...] 1. Le principe donné par M. d’Alembert réduit les lois de la Dynamique à celles de la Statique; mais la recherche de ces dernières lois par les principes ordinaires de l’équilibre du levier, ou de la composition des forces, est souvent longue et pénible. Heureusement il y a un autre principe de Statique plus général, et qui a surtout l’avantage de pouvoir être représenté par une équation analytique, laquelle renferme seule les conditions nécessaires pour l’équilibre d’un système quelconque de puissances. Tel est le principe connu sous la dénomination de *loi des vitesses virtuelles*; on l’énonce ordinairement ainsi : *Quand des puissances se font équilibre, les vitesses des points où elles sont appliquées, estimées suivant la direction de ces puissances, sont en raison inverse de ces mêmes puissances.* Mais ce. principe peut être rendu très-général de la manière suivante.” (Lagrange 1870–3, V, 15; Author’s *italics* style and Capital letters; see also *idem* arguments in Lagrange 1764, 8).

³⁸ Lagrange 1764. Nowadays Lagrange’s equations are called *Symbolic Equation of Dynamics*. In other passages Lagrange try to remark and generalize his conclusions in a note: “IV. Scholie. Le principe de Statique que je viens d’exposer n’est, dans le fond qu’une généralisation de celui qu’on nomme communément le principe des vitesses virtuelles, & qui est reconnu depuis longtem[p]s par les Géomètres pour le principe fondamental de l’équilibre. M. Jean Bernouilli est le premier, que je sache, qui ait envisagé ce principe sous un point de vue général & applicable à toutes les questions de Statique, comme on le peut voir dans la Section IX. de la nouvelle Mécanique de M. Varignon, où cet habile Géomètre, après avoir rapporté, d’après M. Bernouilli, le principe dont il s’agit, fait voir par différentes applications, qu’il conduit aux mêmes conclusions que celui de la composition des forces”. (Lagrange 1764, 6. (Author’s Capital letters)). Source: Google Books – Public domain.

of virtual work alone was not sufficient for founding dynamics. It had to be associated with another principle of dynamics, thanks to D'Alembert. The interpretation of D'Alembert's principle as provided by Lagrange has become classic, although it has little to do with the original interpretation (Fraser 1983): accelerating forces (ma), with their signs reversed, balance applied forces.

[IV ...] the Principle of Statics I am introducing, combined with the Principle of Dynamics due to M. D'Alembert gives a kind of general formula which contains the solution of all problems relative to the motion of bodies.³⁹

In other terms:

$$\sum f_i \cdot \delta u_i - \sum m_i a_i \cdot \delta u_i = 0$$

The principle of virtual work (also known as the principle of least action) has the peculiarity of being expressed by a unique symbolic equation. This is a *variational* equation and can be stated without paying particular attention to the choice of reference systems. Since this is quite automatic, it is possible to avoid solving most of the necessary geometrical problems that arise when writing the equation of motion in the Eulerian style. It should be noted that the virtual work equation allows for an easy solution to the problem of constraints: it is sufficient to use virtual displacements congruent with them. In this way, a static as well as a dynamic problem are solved by means of a kinematic study. This still implies the need for some geometric considerations. Lagrange avoided these with a typical trick in his calculus of variations: the virtual displacements were considered free from any constraints that were added in their analytical form to the *variational* equations. For instance if $f(u_1, u_2, \dots, u_n) = 0$ is a constraint equation, the *variational* problem has the form:

$$\delta W(u_1, u_2, \dots, u_n) + \lambda \delta f(u_1, u_2, \dots, u_n) = 0$$

where W is the virtual work and it is a function of u_1, u_2, \dots, u_n , now called Lagrangian multipliers, to which the meaning of constraint reactions can be associated.

Lagrange perfected his approach in *Mécanique analytique*⁴⁰ (Lagrange 1788), where in the introduction, he emphasized the absence of any geometric considerations.

One will find no figure in this work. The methods I will expose do not require neither constructions nor reasonings of mechanical or geometrical nature, but only algebraic operations which develop regularly and uniformly.⁴¹

In what follows, having present the constraint equations among the coordinates of various bodies, which are given by the nature of bodies, the variation of these variables will be reduced to the smallest number, so that the resulting variations are completely independent each other and absolutely arbitrary. We then will equate to zero the summation of all terms concerned with these variations; and we will have all the equations necessary to find the motion of the system.⁴²

As a final comment it must be noted that although the Lagrangian principle of virtual work is usually associated with the first edition of *Mécanique analytique* (1788), its elements were provided completely in *Recherches sur la libration de la Lune*. The other question is that of the principle justification. To this end, the two *Mécanique* editions (1787, 1811) tried to offer a satisfactory proof that was considered too weak by many scientists of the time, as will be shown further ahead. Evangelista Torricelli (1608–1647), in his *Opera geometrica* (Torricelli 1644) claimed to have established a rational criterion for equilibrium, playing a fundamental role in mechanics and in the history of mechanics (Capecchi and Pisano 2007). It can undoubtedly be considered the origin of the modern statement of the principle of virtual work.

³⁹ Lagrange 1764, 8. (Author's Capital letters).

⁴⁰ This is the original title of the 1788 edition. Next editions (1815, 1853) refers to *Mécanique analytique*.

⁴¹ Lagrange 1788, vi.

⁴² Lagrange 1788, 197.

Two heavy bodies linked together cannot move by themselves unless their common centre of gravity does not descend.⁴³

With regard to Torricelli's principle, one can also consider John Wallis (Wallis 1693) and Pierre Varignon's (1654–1722) (Varignon 1725) assumptions, which allowed rigorous mathematical physics formulation. This was aimed at founding all statics upon an easy geometric principle: the composition of forces. In this sense, it is also alternative to the *principle of virtual work*. Let us note that in his letter to Johann Bernoulli (1667–1748), Varignon also dealt with the concept of *virtual velocities*, as components of *virtual infinitesimal displacements* towards the direction of the forces (Bernoulli J 1742, II). After Bernoulli, the most significant contribution to the development of the *principle of virtual work* is probably thanks to Vincenzo Riccati (1707–1775) who tried to establish it based on simple principles easily accepted by his contemporaries, introducing *Principles of actions* in *Dialogo di Vincenzo Riccati della compagnia di Gesù* (Riccati 1749) and in *De' principi della meccanica* (Riccati 1772).

The Relationships between Physics and Mathematics in Lazare Carnot's Mechanics

Lazare Carnot's mechanics is an operative type of mechanics (Gillispie and Pisano 2014) and presents a strong attitude to Leibniz's ideas,⁴⁴ *theoretical physics must explain facts with facts*. The mathematics introduced is that which is absolutely necessary, adapted to represent a previously established physical argument. The theory is independent from the (physical and mathematical) concept of absolute space, typical of previous predictive mechanics (see Table 1). For example, in Lazare Carnot's works⁴⁵, the solutions for the equations of motion are *velocity* and *quantity of motion* (Gillispie and Pisano 2014, chaps 2–4, 11). The concept of space for Carnot is concerned with the finite volume of a system. Time is also different from typical Newtonian time. For example, for Carnot, time is not absolute and does not have continual variations. It has only one dualistic variation: before and after. In this regard, it can be noted that by avoiding the use of Newtonian absolute time and space, Carnot's science also omits the use of physical quantities as non-finite mathematical variables, which in common theoretical physics are fundamental for the infinitesimal calculation of the variations of certain physical quantities. Therefore, from the very beginning their theories did not contain abstract notions such as absolute space or force-cause. Lazare Carnot's mechanical theory was limited to algebraic and trigonometric equations (because in this theory the types of equations of the invariants of motion are to be solved with velocity only). Lazare Carnot made this attitude clear in his second book, *Principes fondamentaux de l'équilibre et du mouvement* (1803a) when he stated that all scientific (and mathematical) notions can only come from experiments.

⁴³ "Praemittimus. Duo gravia simul coniuncta ex se moveri non-possent, nisi centrum commune gravitatis ipsorum discenda. (Torricelli 1644, Liber primus De motu gravium naturaliter descendendum, 99, line 4". Author's italics). [English translation is mine]; Pisano and Capecchi 2010; Capecchi and Pisano 2010, 2009c; Pisano and Capecchi 2007).

⁴⁴ In particular, one can see the concept of collision (adopted by Lazare Carnot) presented by Leibniz in his *Dynamica de Potentia et Legibus Naturae Corporeae* (Leibniz 1849–1863, II, sectio III, proposition 1–18, pp 488–507) and the early concept of potential energy (*Ivi*, II, sectio I, 435). E.g., Lazare Carnot introduced an advancement of potential energy in his theory of motion applied to machines (Carnot L 1803a, pp 36–38). On the Leibnizian background in Lazare Carnot, one can also see the famous correspondences in 1677 (*Ivi*, VI, 81–106) between Leibniz and Honoré Fabri (also Honoratus Fabrius, 1607–1688). For a first panoramic view on Leibniz and his dynamics, see Pierre Costabel's (1912–1989) works (Costabel 1960). For the most complete (works and letters) series of Leibniz's mathematical writings, see Eberhard Knobloch's VIII edition for "Berlin-Brandenburgische Akademie der Wissenschaften Leibniz-Edition, Reihe VIII" (Leibniz 2009–), Bussotti 2015. Particularly on Leibniz, in the occasion of his anniversary, see also *The Dialogue between Sciences, Philosophy and Engineering. New Historical and Epistemological insights. Homage to Gottfried W. Leibniz 1646-1716* (Pisano, Fichant, Bussotti and Oliveira 2017; Bussotti and Pisano 2017).

⁴⁵ I remark the general scientific and filial project assumed by Lazare and Sadi Carnot (Pisano 2010) on mechanical and heat machines theories (Pisano 2012). As above mentioned, I refer and use recent own material around Lazare Carnot's science (Gillispie and Pisano 2014) under Springer permission as above and below reported.

Table 1. Carnot's physical–mathematical–geometrical approach

Main concepts	Lazare Carnot (1786; 1803a; 1813)
Space and time	No absolute and infinite Newtonian space and time
Physics–mathematics relationship	Only physical
Geometry–physics relationship	Independence from position in space
Basic concepts	Velocity, Quantity of motion, limited mechanical Work
Mathematical approach	No local and infinitesimal variables
Main mathematical-geometrical technique	Geometric motion

To be precise, in Lazare Carnot's words:

Following this idea ["to avoid metaphysical notion of force" and... to use "the theory of communications of motions"⁴⁶] we will soon see, as I previously mentioned, the necessity of turning to the experiment, and that is what I did, without neglecting to support myself with reasonings that can confirm it in the most plausible way, using or generalizing the results per induction. At times I even used the name of the force in the vague sense of which I spoke above [...].⁴⁷

[...] *Primitive ideas concerning the matter, the space, the time, the rest, the motion, etc.* 7. The first rule to establish in such delicate research on the laws of nature is to only admit notions so clear that they can comprise the bounds of our logic. We must therefore reject the definitions of *matter*, *time*, *space*, *rest*, and *motion* as expressions that are impossible to express with more clear terms, and the ideas that these expressions produce in us primitive ideas outside of which it is impossible to construct. But once these expressions are admitted, we will easily see that which is a body, speed, a motive force, etc. 8. The body is a given part of matter. 9. The apparent space that a body occupies is called its *volume*; the actual space that this same body occupies, or its real quantity of matter, is called its *mass*. When the body is such that equal parts of its volume always correspond to equal parts of its mass, we say that it has a uniform *density*, or that it is equally *dense* in all of its parts; and the relationship from mass to volume, or the quotient of one times the other, is called the *density* of this body. But if unequal masses correspond to equal volumes, we say that the density is variable and for each particle of matter, we call *density* the volume of this particle divided by its mass, or rather, the last reason of these two quantities. The empty parts or gaps lodged between the parts of the matter, and that make the volume or apparent space greater than the actual space are called *pores*.⁴⁸

[On the concept of force in the theory]. [...] in my opinion, no rigorous proof of the parallelogram of forces is possible: the mere existence of the *force* in the announcement of the proposition is able to make this demonstration impossible for the nature of things in itself. "It is extremely difficult", as Euler said, "to reason on primary principles of our knowledge [...]". This obscurity disappears in the second way [theory of motion] to conceive the mechanics, but another inconvenience appears; that is the fundamental principles that in the first way [theory of forces where cause produces motion] are established such as axioms in favor of the metaphysical expression [...] that is to say, [...] force, are, in this second case [theory of motion], nothing less than self-evident propositions, and in order to establish them, we need to include the recourse to the experience.⁴⁹

In the Newtonian paradigm until Laplace's physics (Fox 1974; see also Pisano and Gaudiello 2009) and generally speaking for mathematical physics, one can see a strong use of geometry and mathematics (e.g., differential equations). Thus, a mathematical result is obtained.⁵⁰ *But, what about from physical point of view*⁵¹?

⁴⁶ Carnot L 1803a, XVI, line 5.

⁴⁷ Carnot L 1803a, XVI, line 10.

⁴⁸ Carnot L 1803a, 6–7, line 1. (Author's *italics*).

⁴⁹ Carnot L 1803a, xii–xiv, line 17.

⁵⁰ In the 1816 Laplace pointed out that the speed of sound in air depended on the heat capacity ratio and corrected Newton's surprising error (Biot 1858, 1–9, 1802, 173–182).

⁵¹ E.g., the second Newtonian law is not a strictly physical equation. It is – in modern term – a second order differential equation that would interpret (physically) the law of motion. It does so by a mathematical–physical equation, which, of

Newton gave, in the second book of his *Mathematical Principles of natural Philosophy* the expression of the speed of sound: how he achieves this is one of the most remarkable features of his genius.⁵²

When the temperature of the air is raised, at constant pressure, only part of the heat is used to produce that effect [to raise the temperature]: the other part, which becomes latent, serves to increase its volume. This latter part of the heat is liberated when the air is reduced to its primitive volume by an increase in pressure. When two air molecules come close together in a vibration, the heat released raises their temperature and tends to radiate out into the nearby area; but if this happens very slowly relative to the speed of vibration, we can suppose that the amount of heat remains the same [for the two molecules]. Thus, as the two molecules approach, they meet a resistant force, first, because since their temperature being supposed constant, their [forces of] repulsions augment in inverse proportion to their distances; and second because the latent caloric which develops increase their temperature. Newton only considered the first of these causes of repulsion; but it is clear that the second cause must increase the speed of sound, since it increases the pressure. By entering it in the calculation, I come to the following theorem: "The real speed of sound is equal to the product of the Newtonian formula times the square root of the ratio of the specific heat of air at constant pressure of the atmosphere and at different temperatures, to its specific heat at constant volume".⁵³

Newton's calculation gave 968 (920–1085) English feet per second (Newton 1803 [1686-7], 371–372), which is ca. 20% shorter than the value of the speed of sound and later 979 English feet per second appeared (Newton 1714, 343–344). It may have been convenient for the experimental data of the time but it was undoubtedly too low of a value⁵⁴. In effect, the adiabatic compression of the air, which results in a local rise in temperature and pressure, was also taken into account.

Laplace's investigations in practical physics were confined to those carried out jointly with Lavoisier on the specific heat of various bodies from 1782 to 1784. It should also be noted that this is a similar technique to the one Émile Clapeyron would use in 1834 to reformulate Sadi Carnot's theory, but he would not succeed in doing so with his theorem (Clapeyron 1834, 153–190). Lazare Carnot considered the mathematical technique with the differential to be inaccurate but he did not believe in caloric (Carnot L 1990). In fact, he considered infinitesimal analysis (Gillispie 1971, ft. 1, 12, 1979, pp 251–298, § 13, 256) to be a very clear mathematical apparatus in and of itself, which varies with continuity by means of concrete variables. However, for differentiated variables in the previous technique, the mathematical problem is the opposite: the aim is to determine function Q by using an abstract calculation. Therefore, as Lazare Carnot explains in a footnote in *Principes fondamentaux de l'équilibre et du mouvement* (Carnot L 1803a, 11, ft. 1), infinitesimal analysis is not suitable in these cases. A different type of mathematics is necessary in which geometry acquires a greater importance.⁵⁵ Lazare Carnot's mathematics select geometric motions, which by definition, admit their opposites.

Lazare Carnot⁵⁶ is usually considered to be the foremost author who claimed that the empirical nature of mechanics was both theoretical and mechanical (Gillispie and Pisano 2014). Lazare Carnot expressed his view of mechanics in the introductory parts of *Essai sur les machines en général* (Carnot L 1786) and *Principes généraux de l'équilibre et du mouvement* (Carnot L 1803a). This is what Carnot wrote in his *Essai sur les machines en general*:

course, one cannot establish experimentally, as instead can be done by dynamometer to measure magnitudes in a static equation, e.g., Hooke's law (Pisano 2007, 2009a, 2009b).

⁵² Laplace 1816, 238, line 7; author's *italics* and Capital letters.

⁵³ Laplace 1816, pp 238–239, line 24. (Author's quotation marks).

⁵⁴ Finn 1964, ft. 19, 8; Newton 1999, pp 772–778.

⁵⁵ On mathematical and geometrical standpoints see: Chemla 1990, 1998; Nabonnand 2010, 2011.

⁵⁶ For the biography see: Gillispie 1970–1980 III, 70–79. On Lazare Carnot's mechanics see the fundamental Gillispie's works: Gillispie 1971, 1976, Gillispie and Youschkevitch 1979.



Fig. 5. Lazare Carnot's *Essai sur les machines en général*⁵⁸

In the first part (Carnot L 1786, 104–107) Carnot declared his preference for the analytic approach and in the second part he declared the two principles assumed in *Essai sur les machines en général* (the action and reaction and the conservation of *momenta* in the impact) empirical laws. In the other work, for the introduction of *Principes généraux de l'équilibre et du mouvement* he reasserted his empiric idea:

Ancients established as an axiom that *all our ideas come from senses*; and this is no longer object of dispute [...].⁵⁹

⁵⁷ Carnot L 1786, 104–107.

⁵⁸ Carnot L 1786. Source: Google Books – Public domain.

⁵⁹ Carnot L 1803a, 2. Author's *italics*.

But he also expressed the opinion that the laws of mechanics can be considered as either empirical or fully rational:

3. This notwithstanding sciences do not derive in the same way their basis from experience. Pure mathematics derive from them less than all the others; then mathematical physic sciences, then physical sciences. [...].

4. It would be certainly satisfactory, in each science, to be able to decide the point where it breaks off to be experimental and becomes rational; that is to reduce as much as possible the number of possible truths we must obtain from the experience and when accepted are sufficient with the sole reasoning to follows all the branches of the science. But this seems to be too difficult. If one wants to go up too much one will venture to give dark definitions ad vague and scarcely clear proofs. There are less drawbacks to obtain from the experience more items of information than those strictly necessary. [...].

It is thus from experience that men derived the first notions of mechanics. This notwithstanding the fundamental laws of equilibrium and motion [...] appears from one hand so natural to reason, and from the other hand they manifest themselves so clearly by means of the most common facts, that it seems difficult to say that is from one instead than from the other that we derive the complete conviction of these laws.⁶⁰

Carnot specified the role analysis plays in the establishment of these laws, which are referred to as hypotheses:

72. Now it has to establish upon given facts, and upon other observations which we still could have, *some hypotheses* [italic is ours] which are constantly in accord with these observations and which we can assumes as general laws of nature.⁶¹

It is not necessary to have concerned phenomena hypotheses which are unrelated to each other:

73. My objective is not to reduce them [the hypotheses] to the smallest number; it is enough for me that they were consistent and clear enough [...] but they are the most suitable to confirm the principle [the experimental facts], by showing that they are, as to say, nothing but the same truth which says all the same under different forms.⁶²

Carnot assumed seven hypotheses which are summarized in the following Table 2.

Table 2. Some of Carnot's hypotheses (Carnot 1803a)⁶³

1	Once at rest a body cannot move by itself and once put in motion it cannot change neither its velocity nor its direction by itself (<i>Ivi</i> , 49).
3	When many forces, either passive or active, equilibrate themselves, each of this force is always equal and opposite to the resultant of all the others (<i>Ivi</i> , 49).
5	The action that two bodies contiguous exerts each other by impact, pressure or tension, does not depend in any way by their absolute velocity, but only by their relative velocity (<i>Ivi</i> , 49).
7	When bodies who impact are perfectly hard or perfectly soft, the proceed always together after the impact; that is according to the straight line of their mutual action [...] (<i>Ivi</i> , 50).

In modern didactic presentations of classical mechanics, force is considered as varying continuously. This was not the case in the XVIII century where the impact of bodies was also considered very important. The relevance of impact was mainly due to the dominant atomistic conception of matter where impact among atoms was the only way to transmit forces. According to Newton's *Opticks* (Newton 1730) the ultimate

⁶⁰ Carnot L 1803a, 3–5.

⁶¹ Carnot L 1803a, 46–47.

⁶² Carnot L 1803a, 47.

⁶³ Adapted from Pisano and Capecchi 2013, 113.

constituent of matter was small bodies not completely deformable or hard. Due to this property, it was natural to argue on a rational basis that when two hard bodies of equal mass and opposite velocity collide they cannot help but stop. When they touch, they must stop because of their impenetrability and they then remain at rest because there is no reason for a rebound. Some scientists however did not accept the hard body model, and among them were Johann Bernoulli and Euler who agreed with Leibniz (Bussotti 2014, *forthcoming*). In particular Euler divided bodies in more or less soft (*mollioribus corporibus ut cera vel argilla*) and elastic (*elastica*) (Euler 1738 [1730–1], 161). Independently of its foundation any formulation of mechanics had to address and explain global phenomena where variation of force and motion occurs either continuously or through an impact.

On one hand, Euler, who decided to base mechanics on force acting continuously, considered impact as a continuous process: when two bodies collide, they are deformed and exchange forces which continuously vary in time, even if this happens in a very short time interval. The knowledge of the way the matter warps allows for the discovery of the law of (continuous) force the bodies exchange and for the study of the effect of the impact with the law of a mechanics where force is a continuous quantity, even when bodies are extended and not simple material points. On the other hand, D'Alembert and especially Lazare Carnot, who based their mechanics on impact, considered the continuous variation of force to be due to a sequence of infinitesimal impacts.

On Lazare Carnot's Concept of Work

There are two ways to deduce mechanics from its principle. The first is to consider it as the theory of forces, that is the causes which impress motion. The second is to consider mechanics as the theory of motion in itself.⁶⁴

Lazare Carnot preferred the second way. However, he was not against the term *force* which he used quite often, sometimes with a technical meaning: “[...] will call moment of activity, consummated by this force in a given time, the sum of moments of activity consummated by it at every instant [...]”⁶⁵, sometimes following common sense, others even intending the meaning of work. Carnot maintained that as far as the motion of a machine is concerned, force is not the most important concept because the effect it produces also depends on the way it is applied. Carnot used the concept of work to take this way into account (Gillispie and Pisano 2014). He was not the first to do this, but he was the first to emphasize it and give it an operational meaning as a foundation of mechanics, especially for applied mechanics (Gillispie 1976, 1979). The term he used to indicate work was *moment of activity*:

XXXII. If a force P moves with a velocity u and the angle formed by P and u is z , the quantity $P \cos z \, u \, dt$, where dt is the element of time, is called *moment of activity* consumed by force P during dt .⁶⁶

The total moment of activity during a finite interval of time T is given by⁶⁷:

$$P \int u \, dt \cos z$$

Lazare Carnot could quite easily formulate, as a corollary, a fundamental result of his mechanics: the conservation of work:

*Corollary V. Particular law concerning the Machines whose motion changes by imperceptible degrees. X LI. In a Machine whose motion changes by imperceptible degrees, the moment of activity in a time given by soliciting forces, is equal to the moment of activity, exerted at the same time by resistant forces.*⁶⁸

⁶⁴ Carnot L 1803a, xj.

⁶⁵ Carnot L 1786, 65–66. Author's *italics*.

⁶⁶ Carnot L 1786, 65–66. Author's *italics*. See also: *Ivi*, pp 96–97.

⁶⁷ Carnot L 1786, 66.

⁶⁸ *Ivi*, pp 75–76. (Author's *italics*).

Lazare Carnot considered the *production of work* to be produced by mechanical machines⁶⁹ (Oliveira 2014, Chaps. V-VI, 118–171). The f_i -forces are only important when they are linked to δs_i -displacements of bodies. In his father's mechanical theory, the *production of mechanical work* occurs with the transference of motion from one body to another, both being constrained bodies.

Preface. Although the theory here presented is applicable to all issues concerning the communication of motions, *Essay on machines in general* was given as title of this pamphlet; first of all, because they are mainly the Machines that are considered the most important argument of mechanics; secondly, because no particular machine is dealt with but we only deal with properties which are common to all of them.⁷⁰ [...] XXXII. If a force P moves having velocity u , we call z the angle formed by u and P , the quantity $Pudt\cos z$, where dt is the element of time, will be called *moment of activity* consummated by the force P during dt ; that is the *moment of activity* consummated by a force P in an infinitesimally short time, is the product of this force, orientated such as its velocity, and the path that the point, [fds] where this force is applied, does in an infinitesimally short time. I will call *moment of activity*, consummated by this force in a given time, the sum of *moments of activity* consummated by it at every instant [...] ⁷¹. [...] we come back specifically to the second way [theory of motion] of looking at the problem, that is to say, that mechanics are nothing else than the theory of the laws of the communications of the motions.⁷² [...] The first method [theory of forces where cause produces motion] offers much more ease; so it is, as I mentioned here above, almost generally followed. Nevertheless, I adopted the second [theory of motion] as I already did in the first edition; because I wanted to avoid the notion of metaphysics of forces, to leave undistinguished the cause and the effect, in short, to bring everything to the only theory of communication of motions.⁷³

However, it should be noted that the analogy does not go any further.⁷⁴ To be more precise, according to Lazare Carnot, an action of every force with the weight force can be reproduced. Due to the *communication of motion* (Carnot L 1803a, xij–xvj), in the end, one can also theorize on the work it completes as well as the idea *en général* of producing a new physical situation based on the impossibility of perpetual motion. Lazare Carnot studies a mechanical machine in general based on the fundamental affirmation that *perpetual motion is impossible* and the independence from the working substance, bodies and mechanism:

[...] everyone repeats that in Machines in motion time or speed is always lost when force is gained [...].⁷⁵

⁶⁹ On the role played by science and technique/machines within history of science see recently Pisano and Bussotti 2015b, 2015a.

⁷⁰ Carnot L 1786, iij, line 1. Author's *italics*.

⁷¹ Carnot L 1786, 65–66, line 2 Author's *italics*.1; see also pp 96–97. Author's *italics*.

⁷² Carnot L 1803a, xij, line 4. Author's *italics*.

⁷³ Carnot L 1803a, xv–xvj, line 24. Author's *italics*. See also 1803a.

⁷⁴ An analogy between mechanical and heat machines should be noted. If in thermodynamics Q is analogous to f , since neither are state functions f must be substituted by potential $\Delta V=f\Delta s$, while Q must be substituted by entropy, which however has a different formula $\Delta S= \Delta Q/t$. (Thomson 1851, I, 175–183; see also Clausius 1850, vol 155, 368–397; 500–524). Moreover, it should be also noted that in the second case, it is not a special physical distance but it is temperature–range, $\Delta t \neq 0$. Sadi Carnot wrote this at the beginning of the discursive part of *Réflexions sur la puissance motrice du feu* and repeats it several times as well as at the end of the demonstration of his celebrated theorem (Carnot S 1978, 38): *work can be obtained every time there is a difference in temperature between which heat passes*. Thus, it is possible to note a common way of conceiving work in comparison with special and heat motions (Pisano 2010; Gillispie and Pisano 2014).

⁷⁵ [...] tout le monde répète que dans les Machines en mouvement on perd toujours en temps ou en vitesse ce qu'on gagne en force [...]. (Carnot L. 1786, vi, line 14; see also *Ivi*, viii, line 20).

[...] The reflections I propose on this law [lvi, vi, op. cit.] lead me to say something about perpetual motion and I will show not only that every machine which is aborted must stop, but I will assign the very instant when this must occur.⁷⁶

[...] But, I repeat, this Trial only concerns machines in general; each of them have their own particular properties.⁷⁷

[...] we compare these different efforts regarding the agents that produce them, because the nature of the working substance cannot change the forces they must exert to fulfil the different objects for which the Machines are intended.⁷⁸

[...] LVII. what is finally the veritable purpose of moving machines? [...] *the machines in motion, always lose time and velocity, what is they gained in force.*⁷⁹

[...] LXII. We can conclude from that which we have just said regarding friction and other passive forces, that perpetual motion is absolutely impossible, using it to produce only bodies which are not solicited by any motive forces and even heavy bodies [...].⁸⁰

[...] It is therefore evident that we must absolutely give up the hope of producing that which we call perpetual motion if it is true that all of the motive forces that exist in nature [...].⁸¹

Table 3. On the way of conceiving vincula and the production of work

Lazare Carnot (1780; 1786)

The work as a *product* of a mechanical machine; *vincula* bodies.

Mechanical vincula: $M \gg m$

(*Principle of virtual work*). Systems of bodies, non–infinitesimal points, but global and with *vincula*.

More than one body having infinite mass cannot be a machine: no work from *vincula*, only.

It is impossible to link (in a direct way) different potential systems to produce work freely (impossibility of perpetual motion).

108. When a body acts on another one it is always directly or through some intermediary body. This intermediate body is in general what we call a machine. The motion that is lost at every moment in each of the bodies applied to this machine is partly absorbed by the machine itself and partly revised by the other bodies of the system but as it may happen that the subject of the matter is only to find the interplay of the bodies applied to the intermediate bodies without the need to know the effect on the intermediate bodies, we have imagined, in order to simplify the question, to ignore the mass of this body, however keeping all the other properties of matter. Hence the science of machines has become a sort of isolated branch of mechanics in which it is to be considered the mutual interplay of different parts of a system of bodies among which there are some that, lacking the inertia as common to all the parts of the matter as it exists in nature, withheld the names of machines. This abstraction might simplify in special cases where circumstances indicating those

⁷⁶ “Les réflexions que je propose sur cette loi [Carnot L 1786, vi], me conduisent à dire un mot du mouvement perpétuel, & je fais voir non–seulement que toute machine abandonnée à elle–même doit s’arrêter, mais j’assigne l’instant même ou cela doit arriver.” (Carnot L 1786, ix, line 16).

⁷⁷ “Mais, je répété, cet Essai n’a pour objet que les machines en général; chacune d’elles à ses propriétés particulières [...]” (Carnot L 1786, x, line 14). After a re–elaborated work (1781) a publication (nowadays lost) appeared on 1783. The next edition (Carnot L 1786) is the only one consultable.

⁷⁸ “[...] l’on compare ces différents efforts dans égard aux agents qui les produisent, parce que la nature des agents ne peut rien changer aux la nature des agents ne peut rien changer aux forces qu’ils font obliges d’exercer pour remplir les différents objets auxquels sont destinées les Machines [...]” (Carnot L 1786, 62, line 2). After a re–elaborated work (1781) a publication (nowadays lost) appeared on 1783. The next edition (Carnot L 1786) is the only one consultable.

⁷⁹ “LVII. Quel est donc enfin le véritable objet des Machines en mouvement? [...] *les Machines en mouvement, on perd toujours en temps ou en vitesse ce qu’on gagne en force.*” (Carnot L 1786, 88–89, line 24 (Authors’ *Italic style*)).

⁸⁰ “LXII. On peut conclure de ce que nous venons de dire au sujet du frottement & autres forces passives, que le mouvement perpétuel est une chose absolument impossible, en n’employant, pour le produire, que des corps qui ne seroient sollicités par aucune force motrice, & même des corps pesants [...]” (Carnot L 1786, 94, line 16).

⁸¹ “[...] Il est donc évident qu’on doit désespérer absolument de produire ce qu’on appelle un mouvement perpétuel, s’il est vrai que toutes les forces motrices qui existent dans la nature [...]” (Carnot L 1786, 95, line 33).

bodies for whom it was proper to neglect the mass to make it easier for the objective, but we easily know that the theory of machines in general has become much more complicated than before because then this theory was confined in the theory of motion of bodies as they are offered to us by nature, but now it is necessary to consider at the same time two kinds of bodies, one kind as actually existing, the other partially deprived of its natural properties. Now it is clear that the first problem is a special case, since it is more complicated than the other so that by similar hypotheses, we easily find the laws of the equilibrium and of motion in each particular machine such that the lever, the winch, the screw, resulting in a blend of knowledge whose binding can be hardly perceived and only by a kind of analogy; this must necessarily happen as we will resort to the particular figure of each machine to show the property which is common to it and to all the others. Since these properties are the ones we have mainly seen in this first section, it is clear that we will be able to find them only by putting aside the particular forms. So let us start by simplifying the state of the issue by ceasing to consider the system bodies of different natures; finally giving back to machines their inertia it will be easy afterwards to neglect the mass in the result, we will hold the possibility to consider it or not, and therefore the solution of the problem will be general and easier at the same time.⁸²

It should be noted that in Lazare Carnot's theory, it is implicit that many bodies with infinite mass, or constraints alone, do not form a machine (Carnot L 1786, 58–59) and therefore never produce work. Following this analogy one can clearly affirm, with the same reasoning as before, that it is impossible for connecting constraints *in a way only directed at different a t thermostats*. That is to say a machine to run (*produced work*) by letting heat pass without restrictions–dissipations. In other words, the reflection on the old experiment of the exchange between two bodies inside a calorimeter cannot show how work is produced. In fact, to produce work, other intermediary mechanisms are necessary in addition to thermostats in order to adequately utilize the transference of heat between the two temperatures. This is the *second argument ad absurdum* that unites the (implicit) development of the two theories, according to their common model of theory based on a problem.

The principle states that the total virtual work performed by all the forces acting on a system in static equilibrium is zero for a set of infinitesimal *virtual displacements* from equilibrium. The infinitesimal displacements are virtual because they need not be obtained by a displacement that actually occurs in the physical system. The virtual work is the work performed by the virtual displacements, which can be arbitrary and are consistent with the constraints of the system. Its common mathematical expression is:

$$\delta W = \sum_i F_i^{(a)} \delta s_i = 0$$

The theory of mechanical machines may be based on the *principle of virtual work*, and thought of as a consequence of the principle of the impossibility of perpetual motion, e.g., applied to machines and constraints: *it is impossible that the reactions of the constraints on the actions of the bodies, which make up the machine, produce positive work*. In other words, it is impossible for forces of bodies of constraints to produce work:

$$\sum_i R_i ds_i \leq 0$$

On Lazare Carnot's *Principle of Virtual Laws*

Lazare Carnot established an approach to science which was different from the common paradigm of his time (Dhombres and Dhombres 1997). The generalization of Lazare Carnot's *Principle of virtual work* is historically very important because it precedes Lagrange's approach in his *Mécanique analytique* (Lagrange 1788).

Lazare Carnot began by stating his principles, which he also referred to as laws, to underline their empirical content. In this regard, Charles Coulston Gillispie (1918–2015) observed:

⁸² Carnot L 1780, § 108.

He [Lazare Carnot] did achieve a greater clarity, most notably in the passages defining geometric motion: Any motion that, when imparted to a system of bodies, has no effect on the intensity of the actions that they exert or can exert on each other in the course of any other motions imparted to them, will be named geometric [“DÉFINITIONS” (Carnot 1803a, § 136, 108). Neither in the 1780 memoir nor in 837 the *Essai sur les machines en général* had Carnot adapted his concept of geometric motions from the principle of virtual velocities. In the *Principes fondamentaux de l'équilibre et du mouvement*, however, he went on to recognize the analogy between such motions and that principle in the use Lagrange made of the latter⁸³

They are only two principles in *Essai sur les machines en général* which become seven in *Principes généraux de l'équilibre et du mouvement* as we have already seen.

First law: *the reaction is always equal and contrary to action.*

Second law: *when two hard bodies act on each other, because of an impact and pressure, that is because their impenetrability, their relative velocity, immediately after the impact, is always zero.*⁸⁴

The first law states that all bodies which change their state of rest or motion always do so due to the action of another body. All bodies resist their change of state. Referring to this resistance Carnot uses the term “inertia force”, defending himself, for example, from Euler who considered it to be a confusing concept. This was because of the union of the contrasting ideas of activity (force) and passivity (inertia), which still prevailed in applied mechanics. For Carnot, the inertia force was “the result of the present motion and of a motion equal and opposite with respect to that which it must have in the subsequent instant” (Carnot 1786, pp 60–61).

The second law concerns hard (or completely soft) bodies. Carnot was convinced that this law put aside elastic bodies. He declared this openly and justified it by assuming that the behaviour of elastic bodies can be re-conducted to that of hard bodies considering the former as composed of many small hard bodies connected by springs. It is clear that Carnot’s is a forced justification because the way to quantify elasticity has yet to be clarified. By applying his principles to a system of free hard bodies, or to a system of bodies connected by rigid and insensible rods, Carnot obtained a first principle of mechanics which had the following form:

$$\sum mVU\cos Z = 0$$

This was named the “first fundamental equation of mechanics”. Here m is the mass of the corpuscles of the system, V is the true velocity after the impact, U is the lost velocity (such that $W = V + U$ is the velocity the mass would have before the impact) and Z is the angle between V and U . At this point Carnot introduced the concept of geometric motion.

XVI. [...] *if a system of bodies sets out from a given position, with an arbitrary movement, but yet of such [a nature] that it is possible to make it take another in every respect equal and directly opposite; each of these movements will be named a geometrical movement [...]*⁸⁵

⁸³ Gillispie and Pisano 2014, 72, line 23.

⁸⁴ Carnot L 1786, pp 21–22. Author’s *italic*.

⁸⁵ Carnot L 1786, 28. Author’s *italic*. See also *Ivi*, 29–34, 41–45. See also Carnot L 1808a, 212; 1808b.

In *Principes généraux de l'équilibre et du mouvement* the definition is slightly different:

<p>108 PRINCIPES DE L'ÉQUILIBRE</p> <p>de ces résultats à la fin de la première partie. Il ne s'agit ici que de démontrer ces résultats d'une manière rigoureuse par le seul raisonnement, en partant des hypothèses établies ci-dessus, et d'en déduire les conséquences les plus générales. Nous commencerons par le choc des corps, soit immédiat, soit opéré par l'entremise d'une machine. Nous en déduirons ensuite, comme cas particulier, les lois du mouvement d'un système de corps, lorsque ce mouvement change par degrés insensibles. Cette théorie renfermera donc tous les principes fondamentaux de la communication des mouvemens, et par conséquent, de la mécanique elle-même; car, ainsi que nous l'avons déjà observé, on ne considère, en mécanique, aucune force qui ne réside effectivement dans les corps, c'est-à-dire, qui ne soit réellement une quantité de mouvement déjà produite.</p> <p>D É F I N I T I O N S.</p> <p>136. <i>Tout mouvement, qui imprimé à un système de corps ne change rien à l'intensité de l'action qu'ils exercent ou pourroient exercer les uns sur les autres si on leur imprimoit d'autres mouvemens quelconques, sera nommé mouvement géométrique.</i></p> <p><i>La vitesse que prend alors chaque mobile, sera nommée sa vitesse géométrique.</i></p> <p>Ainsi, par exemple, si lorsque deux corps sont</p>	<p>DEFINITIONS.</p> <p>136. <i>Any motion will be called geometric if, when it is impressed upon a system of bodies, it has no effect on the intensity of the actions that they do or can exert on each other when any other motion is impressed upon them.</i>⁸⁶</p>
<p>Fig. 6. Geometric motion in <i>Principes généraux de l'équilibre et du mouvement</i>⁸⁷</p>	

The first definition is purely geometric: geometric motions are reversible motions congruent with constraints. In the second definition there is also a reference to mechanical concepts, because the word “action” calls for concepts like force or work. From the examples Carnot gave in the Essay, it appears that geometric motions can also be infinitesimal (see the example referred to in the note on 26). The same holds true for *Principes généraux de l'équilibre et du mouvement* (see theorem IX, 130). From an operational point of view the finite or infinitesimal nature of geometrical motion makes no difference because what Carnot used is velocity u associated with the geometric motion, called geometric velocity. It can be said that the modern concept closest to geometric motion is that of virtual velocity, which today is not often distinguished from virtual displacement. Carnot gave great emphasis to geometric motions, considering their introduction as one of his major contributions to mechanics:

The theory of *geometric motions* is very important; it is as I have already noted like a mean science between ordinary geometry and mechanics [...] This science has never been treated in details, it is completely to create, and deserves both for its beauty and utility any care by Savan[t]s.⁸⁸

With the aid of geometric motion, the first fundamental equation of motion can be rewritten in a more meaningful and expressive way. It is easy for Carnot to show that the first fundamental equation remains valid when the true velocity is V after the impact is substituted by the geometric velocity u , to obtain:

$$\sum muU\cos z = 0$$

⁸⁶ Carnot L 1803a, 108. Author's *italics*. See also: Gillispie 1971, 43.

⁸⁷ Carnot L 1803a. Source: Google Books – Public domain.

⁸⁸ Carnot L 1803a, 116. Author's *italics*.

Now z is the angle between u and U . Carnot called this equation the “second fundamental equation of mechanics” and noted that by varying u among all the possible geometric motions one can obtain all the equations needed to find the lost motions U of all masses. In this way, Carnot solved what seems to be his main problem: given the initial velocities V of a system of masses to find the final velocities W after the impact. Indeed, when U is known, the final velocities are given simply: $W = U + V$. Unfortunately, this problem has little practical value because one rarely has to address the impact of hard bodies, simply because hard bodies do not exist, not even in an approximate way. To pass from the ideal dynamics of impact to the more realistic dynamics of force varying continuously, Carnot provides a passage which is difficult for us to understand: Carnot passed from the expression of motion lost in the impact to that of motion lost by imperceptible degrees and identifies mU (the lost motion) with the force F , which can now be read with the modern meaning.

The tension of treads, or the pressure of a bar, expresses equally both the effort which is exercised on the machine and the quantity of motion who itself loses because of the reaction he tries: if so one call F this force, this quantity F will be the same thing as what is expressed by mU in our equation.⁸⁹

In addition, we can then write the second fundamental equation “F” (Carnot L 1786, 32) as:

$$\sum Fucosz = 0$$

Which is, in fact, the equation of virtual work as given by Lagrange. Particularly, Lazare Carnot also dealt with the principle of virtual work and by means of *geometric motion* (in modern terms, virtual velocities), canonically formulated the principle of virtual velocities in a fundamental theorem (Carnot L 1786, § XXXIV, 68–69). In effect, since his theory of geometric motions coincided with velocities and not with displacements, this allowed Lazare Carnot to avoid, in the formulation of the principle of virtual work, infinitesimal displacements, which could have produced some scientific embarrassment with respect to his assumptions (Carnot L 1813). Furthermore, for the principle of virtual velocity related to any (general) mechanical machine, one can claim that the (forces–) weights that balance each other are reciprocal to their virtual velocities. Incidentally, the two conceptually different approaches/formulations can be mathematical equivalents using the concept of virtual motion as key reasoning.

Lazare Carnot formulated the principle of virtual work by beginning with his law of collisions (Carnot L 1786, 1803a) and without (generally speaking) using classical Newtonian forces. Particularly, Lazare Carnot used the principle of virtual work to discuss and define the conditions of equilibrium of the forces applied to the bodies.

General principle equilibrium and of motion in machines

XXXIV. *Whatever is the state of repose or of motion in which any given system of forces applied to a Machine, exists, if we take it all at once assume any given geometric motion, without changing these forces in any respect, the sum of the products each of them, by the velocity which the point at which it is applied will have in the first instant, estimated in the direction of this force, will be equal to zero.* That is to say, by calling F each of these forces (l), u the velocity which the point where it is applied will have at first instant, if we make the Machine assume a geometric motion, and z the angle comprehended between the directions of F and of u , it must prove that we shall have for the whole system $[\sum]Fucosz=0$. Now this equation is precisely the equation (AA) $[\sum]Fucosz=0$ (Carnot L 1786, 63, line 15)] found (XXX) [lvi , 60] which is nothing else in the end but the same [second] fundamental equation (F) $[\sum]muUcosz=0$ (lvi , 32, line 6)] presented under another form. It is easy to perceive that this general principle is, properly speaking, nothing else than that *Descartes*, to which a sufficient extension is to be given, in order that it may contain not only all the conditions of the equilibrium between two forces, but also all those of equilibrium and of motion, in a system

⁸⁹ Carnot 1786, 65–66.

composed of any number of powers: thus the first consequence of this theorem will be the principle of *Descartes*, rendered complete by the conditions which we have seen were waiting in it (V)⁹⁰.

The aim was to obtain a mathematical expression of its invariant, or the efficiency of a heat machine with respect to all possible kinds of working substances. Therefore, it was necessary to obtain invariants with regard to the efficiency and reversibility of a mechanical machine.

We will now note that in the traditional mechanical theory of hard bodies, the *principle of virtual work* formally defines the condition of *equilibrium* of the forces that act on the bodies in order to produce work:

*Corollary II. General principle of equilibrium in weighing Machines. XXXVI. When several weights applied to any given Machine, mutually form an equilibrium, if we make this Machine assume any geometric motion, the velocity of the centre of gravity of the system, estimated in the vertical direction, will be null at the first instant.*⁹¹

Lazare Carnot, having the mathematical formula for the *principle of virtual work*, studies the theoretical conditions that translate the practical conditions of equilibrium and also obtains his invariants with regard to the efficiency and reversibility of mechanical machines.

His *first* (Carnot L 1786, 32) and *second* equations (Ivi, 33) generalized for multi-body systems are:

$\sum muU \cos(\langle \vec{U}, \vec{u} \rangle) = 0$	(E)	m = mass of the body
$\sum mVU \cos(\langle \vec{U}, \vec{V} \rangle) = 0$	(F)	W = velocity before interaction
		V = velocity after interaction
		$U = W - V$
		u = arbitrary geometric motion

In short⁹²:

- The mass of the parts of a machine.
- Global magnitudes, abstracting from the mass of the mechanism.
- Kinematics first, then dynamics, and statics is a special case of dynamics.
- A theory of machines concerns a theory of the *communication of motions*.
- A machine is a connected system of (hard) bodies.
- The connections between the bodies constrain the *communication* of motion of the bodies.
- The theory of interaction–collisions by means of insensible degrees (e.g., see Carnot L 1803a, § 293, pp 261–262) as the result of a sequence of infinitesimally small percussions.

In order to complete our discussion on the use of the *principle of virtual work* in the two Carnots' theories, in the following section we summarize Lazare Carnot's reasoning upon his laws of conservation, mainly included in both *Essai sur les machines en général* (Carnot L 1786) and *Principes fondamentaux de l'équilibre et du mouvement* (Carnot L 1803a).

⁹⁰ Carnot 1786, § XXXIV, 68–69 and footnote "(I)". (Author's *italics* and Capital letters).

⁹¹ Carnot L 1786, 71, line 1. (Author's *italics*). See also Carnot L 1803a.

⁹² An extensive discussion is in: Gillispie and Pisano 2014.

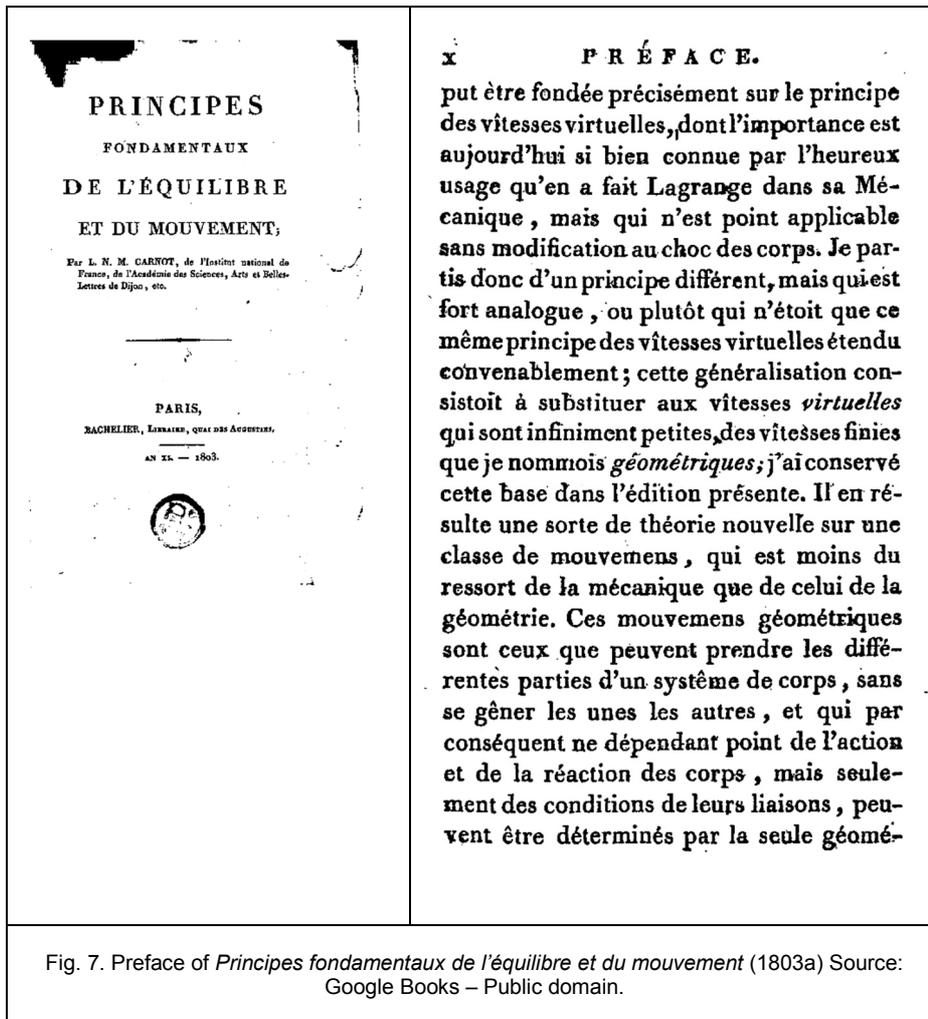


Fig. 7. Preface of *Principes fondamentaux de l'équilibre et du mouvement* (1803a) Source: Google Books – Public domain.

A law of conservation for plastic bodies – even though he called them *hard bodies* – can be written in the following way:

$$\sum_i m \vec{U}_i = 0$$

m_i = mass of i -th body [for isolate system]
 U_i = velocity lost (by that body) during the collision
 W_i = velocity before interaction
 V_i = velocity after interaction
 $\vec{U}_i = \vec{W}_i - \vec{V}_i$

Therefore, using the hypotheses of *parfaitement élastiques*⁹³ bodies (Carnot L 1803a, 105), one obtains:

$$\sum_i m_i \vec{U}_i \vec{V}_i = 0$$

V_i = velocity after interaction is the same for all of them
 Generalization for all bodies using a n -elasticity index.

Now, by using certain calculations, the *law of conservation of kinetic energy* for soft bodies is obtained:

$$\sum_i m_i \vec{W}_i^2 = \sum_i m_i V_i^2 = 0$$

⁹³ Lazare Carnot proposed a generalization from plastic bodies to all bodies by means of an *ad hoc* index. (See below). He presented it in *Essai sur les machines en général* (Carnot L 1786 pp 15–22) and in *Principes fondamentaux de l'équilibre et du mouvement* (Carnot L 1803a, pp 103–106, pp 131–146). The generalization is differently presented in the two cited books. More specifically, the first reports an inverse procedure with respect to second.

Following Lazare Carnot's reasoning presented in both his books (Carnot L 1786; 1803a), at this point, by introducing *geometric motions* (Carnot 1786, 28–30) starting from the previous *law of conservation*, one can write:

$$\sum_i m_i \vec{U}_i \cdot \vec{u}_i = 0$$

m_i = mass of the i -th body
 U_i = velocity lost (by that body) during the collision
 u_i = velocity called "mouvement géométrique"⁹⁴

Here, it is unproblematic to recognize the extension of the *principle of virtual velocity* to the collision of several bodies using

$$\vec{u}_i = \text{const.}$$

With following the proof, which we omit for the sake of brevity, we can write:

$$\sum_i m_i \vec{U}_i \cdot \vec{u}_i = 0; \rightarrow \vec{u} \sum_i m_i \vec{U}_i = 0$$

By considering the u -arbitrariness, we can write:

$$\sum_i m_i \vec{U}_i = 0$$

With

$$\vec{U}_i = \vec{W}_i - \vec{V}_i$$

In short:

- A theory of interacting bodies by means of collisions.
- A collision is a basic phenomenon. In particular, continuously accelerated motion is obtained as a limiting case of a system driven by a series of pulses.
- Newton's second law is replaced by Lazare Carnot's second fundamental equation for a system of n -bodies.
- Due to the arbitrariness of u_i , it can be assumed constant, that is to say the same translation of geometric uniform motions of all bodies is adopted.

By considering another *ad hoc* geometric motion,

$$\vec{u}_i = \vec{\omega} \times \vec{r}_i$$

e.g., the rotation of the system with angular velocity around a fixed axis, and using the properties of the triple product and the arbitrariness of " $\vec{\omega}$ ", he then wrote his two main *laws of conservation* as *invariants of motion*:

⁹⁴ In practice, u is physically the velocity of any *geometric motion*. However, it is a mathematically indeterminate variable and each specification produces (in the equation cited in the running text) an equation applicable to the physical system considered.

$$\sum_i m_i \vec{W}_i = \sum_i m_i \vec{V}_i \quad \text{Law of conservation of the total-quantity-of-motion}$$

$$\sum_i m_i \vec{r}_i \times \vec{W}_i = \sum_i m_i \vec{r}_i \times \vec{V}_i \quad \text{Law of conservation of the total-angular-momentum}$$

Concluding Remarks

The role played by the *Principle of virtual laws* within *classical mechanics* is not easily definable and theoretically it does not appear to be crucial (from a technical standpoint). One can think of *rational mechanics* while avoiding problems⁹⁵ and not allowing for the theory to take on unresolvable problems. In the mechanics applied to rigid bodies, that is, the relationship between physics and its mathematical interpretation, the principle of virtual laws can be used powerfully to solve certain *vincula* problems. Therefore, generally speaking, in modern times it does not appear critical within scientific research. The *vincula* are taken into account by introducing the *constraint reactions* such as *auxiliary unknowns* that are then eliminated by substitution during the solution of the single static problems; and with friction constraints, one can easily provide adequate *constitutive laws*. In *continuum mechanics*, its role is very important since the solution is simpler if one adopts particular values of the mathematical expression of the *principle of virtual laws*, such as using the *method of finite elements*. From a historical point of view it is of utmost importance. It had a long historical process beginning with ancient Greek science (Pisano and Capecchi 2015, chap. 2) until the Aristotelian⁹⁶ and Archimedean mechanical approaches.

Based on some of Lazare Carnot's crucial discussions (Carnot 1786, 28–30), the *geometric motion*⁹⁷ essentially expresses a non-mechanical interaction. Lazare Carnot defined these motions as *invertible*: *a motion assigned to a physical system of interacting bodies is geometric if the opposite motion is also possible*. The result is a *possible motion*, but it is not always *invertible* (e.g., the motion of a sliding ring on a rotating rod). Therefore, one should add the hypothesis of *invertibility* for obtaining the concept of *geometric motion*. Conversely, a *geometric motion*, when integrated, gives an *invertible motion*. At this point, for *vincula* independent of time, a *geometric displacement* is equivalent to a *virtual invertible displacement* (but not vice versa). On the contrary, only a *possible displacement* that is *invertible*, produces, after its derivative, a *geometric motion*. In this sense, we note that initially, the geometric motion is a kind of uniform motion moving on the whole physical system when one considers the equivalence of the state of rest and the state of uniform motion. Consequently, by using double negative sentences (Cf. Gillispie and Pisano 2014, chap. 7) one can write:

$$\neg[(v = 0) \neq (v \neq 0)]$$

Finally, the previous discussion on the *principle of virtual work* and in Lazare's work is, in any case, limited to the two different mathematical approaches with respect to adopted physics. Lazare Carnot himself introduced the reader to a new level of describing physical phenomena (Carnot L 1803a, x), that is to say, the *principle of virtual work*. In this sense, a new kind mechanics – with respect to Newtonian mechanics – would be born thanks to *l'organisateur de la victoire*.

⁹⁵ By the way many assumptions and equations derived by it (i.e. Lagrange, Hamilton, Hamilton–Jacobi).

⁹⁶ In the history of mechanics, if one avoids the theory of centres of gravity (Pisano 2007), a modern interpretation of the principle has its general roots into Aristotelian mechanical school (Pisano and Capecchi 2015, chap. 2).

⁹⁷ Carnot L 1786, pp 28–34, pp 41–45; see also Carnot L 1780, § 113; Gillispie 1971, Appendix C, § 113, 308–309.

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