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What Synergy between Mathematics and Physics is Feasible or Imaginable at Different Level of Education?

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Abstract:
For interdisciplinarity between physics and mathematics to take its proper place in secondary schools, its value must be demonstrated and used during the future teacher's university education. We have observed from examples and surveys, however, that an ever-widening gulf is emerging between degree courses in mathematics and physics. This article therefore develops comparative approaches to some common concepts to demonstrate their complementarities from the angle of the relation between mechanics and analysis. The example of the differential, which is described as an obstacle to the "mathematization" of physics (Karam et al. 2015), is used to transform it into a tool to aid conceptualization, offering a dual approach to the concepts and their applications. This should enable students to have a better understanding that is related to their specific curriculum.

Keywords:
Velocity; Derivative; Differential; Framework; Inference

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Introduction
Historically, the relationship between mathematics and physics has supported the development of both disciplines and fostered the emergence of new theories. This was the case in the seventeenth and eighteenth centuries with the mutual development of analysis and mechanics thanks to the birth of the infinitesimal calculus, dedicated to the nineteenth

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century with the appearance of the notion of limit. René Dugas (1950) describes the seventeenth century as the great creative century with the three peaks that constitute the works of Galileo, Huygens and Newton. He omits Leibniz, but he has a vital contribution to the development of Calculus in Europe. Leibniz also develops the mathematical formalism as Florian Cajori says: "Perhaps no mathematician has seen more clearly than Leibniz the importance of good notations in mathematics" (Cajori 2007, 181).

The book written by R. Courant and H. Robbins confirms the importance of notations:

> Much of the success was due to the marvelous symbolic notation invented by Leibniz. His achievement is in no way diminished by the fact that it was linked with hazy and untenable ideas which are apt to perpetuate a lack of precise understanding in minds that prefer mysticism to clarity. (Courant and Robbins 2015, 443)

Nicolas Bourbaki confirms these subjects:

> Very young, too, he had conceived another idea much more original, that of the usefulness of symbolic notations as "thread of Ariadne" of the thought: "the true method" he says, "we must provide a filum Ariadnes, that is to say a certain sensible and gross means, which leads the mind, as are the lines drawn in geometry and the forms of operations that are prescribed to apprentices in Arithmetic. Without this our mind cannot go a long way without going astray". (Bourbaki 1984, 16)

Bourbaki indicates further in the book that the seventeenth is that of the Calculus.

> It is thus almost exactly within an interval of a century that the infinitesimal calculus has been forged, or, as the English have come to say, the Calculus par excellence; and nearly three centuries of constant use have not yet completely blunted this incomparable instrument. (Bourbaki 1984, 207)

In the eighteenth century, mechanics developed, in particular kinematics and dynamics under the impulsion of Newton and Leibniz, in parallel with mathematical analysis. Dugas indicates in this connection:

> The third part is devoted to the Eighteenth Century, which emerges as the century of the organization of mechanics and finds its climax in the work of Lagrange, immediately preceded by Euler and d'Alembert. The development of mathematical analysis enabled mechanics to take a form which, for a considerable time, appeared to be finally established, and which is still a part of the classical teaching. (Dugas 1950, 12-13)

After the formalism printed in the seventeenth century, mathematical rigor is gradually being put in place, according to Bourbaki,

> The theme of mathematical rigor, contrasting with that of the infinitely small, indivisible, differential [...] But the moment was already more to pour the new wine in old skins. In all this we know today, it is the notion of limit that was elaborated; and, if we can extract from Pascal, from Newton, still others, statements which seem close to our modern definitions, it is only necessary to put them in their context to see the invincible obstacles which oppose a rigorous presentation. When, from the eighteenth century, mathematicians, devoted to clarity, wished to place some order in the

> confused mass of their riches, such indications, found in the writings of their predecessors, were precious to them; when d'Alembert for example explains that there is nothing else in differentiation than the notion of limit, and defines it precisely, one can believe that it was guided by Newton's considerations on the "first and last reasons for vanishing quantities". But as long as it concerns only the seventeenth century, it must be admitted that the way is open to modern analysis only when Newton and Leibniz, turning their backs on the past, agree to provisionally seek the justification for new methods, not in rigorous demonstrations, but in the fruitfulness and coherence of the results. (Bourbaki 1984, 215 and 217)

It was the nineteenth century that saw the birth of mathematical rigor through Baron Cauchy. Due to the acceleration of scientific and technological development in the twentieth century, an obligation to specialize is gradually being established, given the extent of new fields. It becomes difficult to find great scientists able to dominate all scientific and philosophical knowledge, which we will see later is detrimental to physico-mathematical collaboration. However, in 1830 already, Auguste Comte (1830,27-29) evoked the dangers of an increased specialization, which will become the rule in the twentieth as indicated by Bachelard (1951), according to the translation of Martineau of Comte,

> In the primitive state of human knowledge there is no regular division of intellectual labour. Every student cultivates all the sciences. As knowledge accrues, the sciences part off; and students devote themselves each to some one branch. It is owing to this division of employment, and concentration of whole minds upon a single department, that science has made so prodigious an advance in modern times; and the perfection of this division is one of the most important characteristics of the Positive philosophy. But, while admitting all the merits of this change, we cannot be blind to the eminent disadvantages which arise from the limitation of minds to particular study. It is inevitable that each should be possessed with exclusive notions, and be therefore incapable of the general superiority of ancient students, who actually owed that general superiority to the inferiority of their knowledge. We must consider whether the evil can be avoided without losing the good of the modern arrangement; for the evil is becoming urgent. We all acknowledge that the divisions established for the convenience of scientific pursuit are radically artificial; and yet there are very few who can embrace in idea the whole of and one science: each science moreover being itself only a part of a great whole. Almost everyone is busy about his own particular section, without much thought about its relation to the general system of positive knowledge. We must not be blind to the evil, nor slow in seeking a remedy. We must not forget that this is the weak side of the positive philosophy, by which it may yet be attacked, with some hope of success, by the adherents of the theological and metaphysical systems. (Martineau 2000, 34)

Still according to the translation of Martineau, Comte (1830, 30) proposed a solution to solve this problem of drift from the specialization, the specialization however necessary for the development of sciences.

> As to the remedy, it certainly does not lie in a return to the ancient confusion of pursuits, which would be mere retrogression, if it were possible, which it is not. It lies in perfecting the division of employments itself, – in carrying it one degree higher, – in constituting one restore speciality from the study of scientific generalities. Let us have a new class of students, suitably prepared, whose business it shall be to take the

> respective sciences as they are, determine the spirit of each, ascertain their relations and mutual connection, and reduce their respective principles to the smallest number of general principles, in conformity with the fundamental rules of the Positive Method. At the same time, let other students be prepared for their special pursuit by an education which recognizes the whole scope of positive science, so as to profit by the labours of the students of generalities, and so as to correct reciprocally, under that guidance, the results obtained by each. We see some approach already to this arrangement. Once established, there would be nothing to apprehend from any extent of division of employments. When we once have a class of learned men, at the disposal of all others, whose business it shall be to connect each new discovery with the general system, we may dismiss all fear of the great whole being lost sight of in the pursuit of the details of knowledge. The organization of scientific research will then be complete; and it will henceforth have occasion only to extend its development, and not to change its character. After all, the formation of such a new class as is proposed would be merely an extension of the principle which has created all the classes we have. While science was narrow, there was only one class: as it expanded, more were instituted. With a further advance a fresh need arises, and this new class will be the result. (Martineau 2000, 34-35)

In the twentieth century, specialization becomes the rule and applied rationalism that allows the emergence of mathematical models generating the development of technologies to develop experiments validating the predictions of the latter. Louis de Broglie summarizes very well the evolution of mechanics in his preface to the work of René Dugas (1950) with the advent of two new mechanics on the fringes of classical mechanics and which will acquire during the twentieth an immense importance and a great development always on the move.

> As is well known, although the principles of statics had been correctly presented by the old scholars those of dynamics, obscured by the false conceptions of the Aristotelian school, did not begin to see light until the end of the Middle Ages and the beginning of the modern era. Then came the rapid development of mechanics due to the memorable work of Kepler, Galileo, Descartes, Huygens and Newton; the codification of its laws by such men as Euler, Lagrange and Laplace; and the tremendous development of its various branches and the endlessly increasing number of applications in the nineteenth and twentieth centuries. The principles of mechanics were brought to such a high degree of perfection that fifty years ago it was believed that their development was practically complete. It was then that there appeared, in succession, two very unexpected developments of classical mechanics on the one hand, relativistic mechanics and on the other, quantum and wave mechanics. (de Broglie as cited in Dugas 1950, 7-8)

Today, however, specialization is giving rise to a different approach to common concepts within university degree courses in mathematics, physics and engineering. This difference is becoming increasingly marked in the teaching of mathematics and physics. In Belgium’s French-speaking universities, we have observed from interviews with lecturers that communication difficulties exist between mathematicians, physicists and engineers. These difficulties are resulting in the creation of separate first-year courses in mathematics or physics for future mathematicians and physicists, with some receiving a mathematics course given by a physicist while others take a course in physics for engineers, which begins with a presentation of the mathematics that will be useful for the course and subsequently makes no reference to the mathematics course. This difference in language and the relationships between the two disciplines are summarized in a journal published by Springer (Karam et al. 2015).

The following is taken from the introduction:

> Since their beginnings in the ancient world, physics (natural philosophy) and mathematics have been deeply interrelated, and this mutual influence has played an essential role in both their developments as illustrated in the quotations above. However, the image typically found in educational contexts is often quite different. In physics education, it is usual to find mathematics being seen as a mere tool to describe and calculate, whereas in mathematics education, physics is commonly viewed as a possible context for the application of mathematical concepts that were previously defined abstractly. This dichotomy creates significant learning problems for the students […] This problem demands a systematic research effort from experts in different fields, especially the ones who aim at informing educational practices by reflecting on historical, philosophical and sociological aspects of scientific knowledge. (Karam et al. 2015, 487).

Furthermore, since the late 1990s, a large number of studies have addressed the disaffection among students with scientific courses that are seen as difficult and demanding, particularly in mathematics and physics. These studies have called for changes in the way science is taught, including removal of the barriers between disciplines, which was described as desirable in May 2006 in an information report entitled the Rolland Report. We cite below a paragraph from the book by Claude Allègre (1995), the former French Minister of Education but also a physicist, who is very critical of the emphasis on mathematics and the place that is given to it in science education:

> This fact has not, however, been incorporated into science education in France. Logical, deductive reasoning is presented as the archetype of science; mathematics is the discipline that is emblematic of this. This has been the dominant view since the 19th century and its prevalence is still increasing. It has gradually established "pure" mathematics as a way of selecting our elites, not only the students in the most prestigious universities, but also and more surprisingly, our doctors and through the "C" classes in secondary schools, all our "good students" even in arts subjects! (Allègre 1995, 390)

This is a regrettable narrative and only serves to add fuel to the fire. Nevertheless, it does reveal a prevailing way of thinking, as was recently highlighted at an informal meeting entitled *Donnons le goût des mathématiques et des sciences* (*Increasing enthusiasm for mathematics and science*), which was held in November 2015 at Louvain-la-Neuve. This meeting brought together representatives from the business world, higher education institutions and secondary school teachers' associations to discuss scientific and engineering topics. One head of engineering stated that in secondary education, mathematics now plays a role in the selection process that is similar to that of Latin in the past. This is a very archaic view of mathematics.

Surely it is now time to seek to bring the two disciplines, mathematics and physics, closer together and also to unite their respective advantages rather than further entrenching the mutual lack of understanding between them instead of reinforcing their mutual incomprehension, by inspiring, on the other hand, examples developed in the seventeenth and eighteenth centuries, but avoiding the pitfall of a false simplification through non-standard analysis?

Two further excerpts below demonstrate the lack of understanding between mathematicians and physicists at the university level who are training new teachers.

The first comes from a mathematician, Jean Dieudonné (1906-1992), who was a member of the Bourbaki group. In 1980 he presented a paper entitled *De la communication entre mathématiciens et physiciens.* He accounts for the lack of understanding on the grounds of increased specialization and the fact that the overall perspective on both areas has been lost.

> This symbiosis was also facilitated by the fact that, from Newton to von Neumann, the men who were constructing mathematical algorithms were also actively engaged in applying them to nature. These men, however, have hardly any successors in our era; this is undoubtedly the source of the growing lack of understanding that is currently seen between physicists and mathematicians. (Dieudonné 1980, 329)

The second excerpt sets out the view of physicist Jean-Marc Lévy-Leblond (1940 - …) taken from one of his articles entitled *Mathématique de mathématiciens et mathématiques de physiciens,* which was published in 1991.

> What was the point of my initial rhetorical warning, when I spoke about mathematicians as seen by a physicist? It is a reaction to a view of mathematics as a discipline that most mathematicians have held for a very long time. I am delighted that this image is changing, but I would say that the burden of proof lies with you, to some extent, and not only in your own practice as mathematicians. For although things may have changed a great deal in research in the last ten years or so, there is one place where there has been little change, and that is in education. (Lévy-Leblond 1991, 46)

This article is entirely in line with our research on interdisciplinarity between physics and mathematics and the obstacles to implementing it in secondary schools. How can interdisciplinarity be promoted if the teachers who are supposed to be applying it were not made aware of it during their university training?

We look at ways of expressing each obstacle in terms of the objective of overcoming it, with the aim of providing guidance on the pedagogical methods used by teachers. This is known as an obstacle-objective (Martinand 1986). At the same time, students need to be shown the benefit of using an interdisciplinary approach. To do this, we use examples of possible dual approaches by physicists and mathematicians, providing two ways of introducing concepts or resolving problems, to avoid the situation where students fail to understand but an explanation is simply repeated through a change of framework (Douady 1984 and 1992).

That is why this article are addressed the comparative study of the approaches to concepts, problems and applications used by mathematicians and physicists. This study is based on the relationship between mechanics and analysis. It is important for future teachers to be aware of different approaches during their training so that they can use them in teaching practice by introducing changes of framework, allowing students to understand the problems better.

For dual approaches, dualities inspired by Gaston Bachelard (1949), we rely on the difficulties encountered at the birth of the physico-mathematical from Mersenne and on the taking flight of Calculus. Great erudites such as Galileo, Euler, Leibniz, Bernoulli, Lagrange, Cauchy […] studied mathematics and physics in synergy. Analogies between past and present difficulties would provide an excellent source of inspiration for constructing different approaches to theoretical or practical problems.

We begin with epistemological dualities around the concepts of instantaneous velocity and derivative, highlighting two complementary frameworks, one algebraic and the other analytical. The debate over which notation is used for the derivative, either Lagrange's or Leibniz's, which forms an obstacle in conceptual or corporative terms, is thereby transformed into a pedagogical tool that provides a dual approach to problems: an applied duality.

A theoretical duality sheds light on deductive and inductive inference (Barth 1987). The former, which is used by mathematicians, involves a thought process that moves from the general or theoretical to the particular or towards a new theoretical truth; the latter, which is more often used by physicists, sets out from a specific observation to produce a conjectured generality.

We end with the use of duality in modelling. The study of the simple pendulum illustrates the concept of an obstacle, by avoiding the use of ordinary differential equations (ODEs) even though Newton's law of mechanics would point to using them.

Finally, the key concepts are defined in a glossary at the end of this article.

Epistemological duality

A conference, entitled *Outils mathématiques dans les cours de physique*, presented at a science conference in Namur in 2011 turned into a heated debate between the speaker (a mathematics teacher) and the delegates, most of whom were physics teachers. Here is the summary written by the two teachers, Zanotto and Looze, of the conference:

> The speech of scientific colleagues challenges us when they describe the difficulties encountered by students who use mathematical tools involved in the resolution of exercises specific to their discipline. Is a mathematical concept approached in the same way during a mathematics class and during a physics course? The intervention of a professor of physics and a professor of mathematics will allow us to bring elements of answers to this question. Course sequences intimately mixing mathematics and physics will be presented. Mathematical objects such as vectors, derivatives, graphs, trigonometry, second-degree functions and logarithms will be contextualized in a very specific context of physics. These situations will allow physics teachers to perceive the gap imagined by students between these two disciplines. Avenues to reduce this distance and thus facilitate the transfer of the notions taught from one discipline to another will be considered.

The physicists accused the mathematician of using Lagrange's notation (f') for the derivative when Leibniz's notation $\left(\frac{df}{dt}\right)$ is more useful for their classes. This debate is not new; in fact, it dates back to the discovery of differential calculus. It demonstrates the difficulty of using the concept of differentials, a significant source of tension between mathematicians and physicists, as Michèle Artigue shows:

> In this second part of our discussion, we will analyze the evolution of the collaboration between mathematics and physics and the problems it has raised using the theme of differentials. Why has this been chosen? Because this was the theme around which the difficulties in cooperation crystallized during the first year. (Artigue 1981, 26)

In the afternoon, Math Tools in Physics Class – Implementation of the morning lecture, the difficulties encountered during the conference were reinforced during the activity. An example of a situation-problem of the mathematics course has generated many reactions from physics teachers. This situation was based on a third-degree equation describing the motion of an object. The teachers asked the mathematician what motion was described by this equation and how to obtain this motion.

Let's quote an extract from Lagrange (1797):

> The simplest motion, after the one we have just considered, would be the one in which we would have $x = c\,t^3$, but nature does not offer us any simple motion of this kind, and we do not know what the coefficient c could represent, by considering it in an absolute way independent of the velocities and forces. (Lagrange 1797, 225)

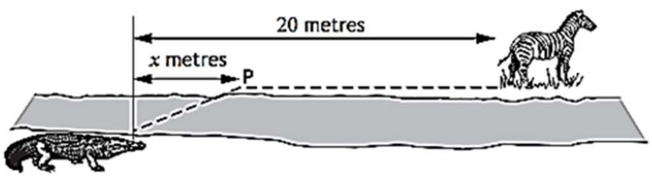
For physicists, it was therefore a statement built for the cause unrelated to physics. Similarly, another difficulty was in mathematics functions, for example, to locate their teaching in the school path, which demonstrates the lack of knowledge of programs in the other discipline. Physics teachers complained about the difficulty students (± 15 years old) face in the functions and their representation in module 1 of the physics course, *Approche expérimentale de fonctions de type y = a.x et d'une fonction non linéaire.* Some functions were not yet seen in mathematics, so teachers reported that it was impossible for students to find the relationships between the variables. The objective set by the program is to check the proportionality or not between the variables, but not to establish formulas that sometimes do not exist. However, the linearization proposed by the program may induce the impression of a need to find the relationship between the variables.

Let's end with the example of a mathematics assessment in Scotland that caused a lot of reactions because of the collapse of students in the face of a physical problem.

A crocodile is stalking prey located 20 metres further upstream on the opposite bank of a river.

Crocodiles travel at different speeds on land and in water.

The time taken for the crocodile to reach its prey can be minimised if it swims to a particular point, P, x metres upstream on the other side of the river as shown in the diagram.



The time taken, T, measured in tenths of a second, is given by

$$T(x) = 5\sqrt{36 + x^2} + 4(20 - x)$$

(a) (i) Calculate the time taken if the crocodile does not travel on land.

(ii) Calculate the time taken if the crocodile swims the shortest distance possible.

(b) Between these two extremes there is one value of x which minimises the time taken. Find this value of x and hence calculate the minimum possible time.

Fig. 1 This math problem that traumatizes Scottish high school students. By Figaro Student, published on 14/10/2015, Exercise "crocodile and zebra", proposed during an examination, did not leave the students indifferent, to the point that some left crying. Faced with the outcry of high school students and their parents, the percentage of correct answers needed to pass the test has been lowered.

These examples prove the lack of collaboration between mathematics and physics at many levels in education. Therefore, it is useful to construct dualities (function-motion, derivative-instantaneous velocity, ordinary differential equation-fundamental law of mechanics) recalling the historical connections at the heart of the Calculus.

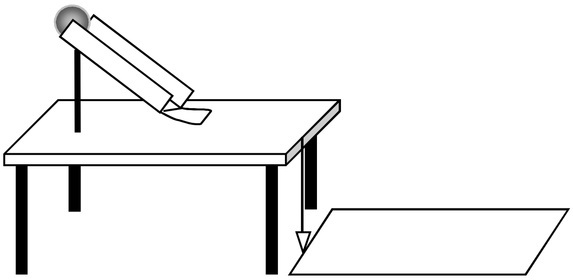
a) Function – Motion: A World of Relationships

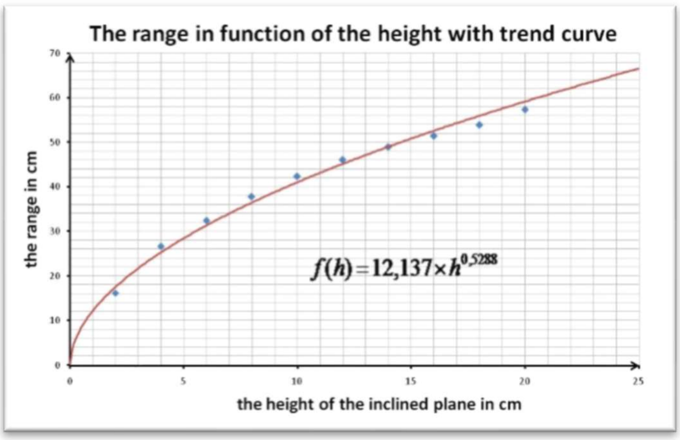
Let's go back to the problem raised by Module 1 and the setting up of laboratories for the experimental approach to functions.

We propose a laboratory statement:

Show if there is a relationship of proportionality between the range in function of the height of an inclined plane.

- Carry out the assembly below.
- Place the angle iron at 2 cm in height.
- Remove the ball at the top and let go.
- Mark on the sheets of paper aligned and attached to the ground the impact of the ball.
- Measure the range.
- Repeat the manipulation of 2 in 2 cm until 20 cm.

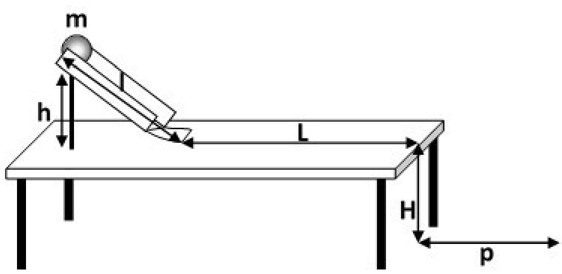


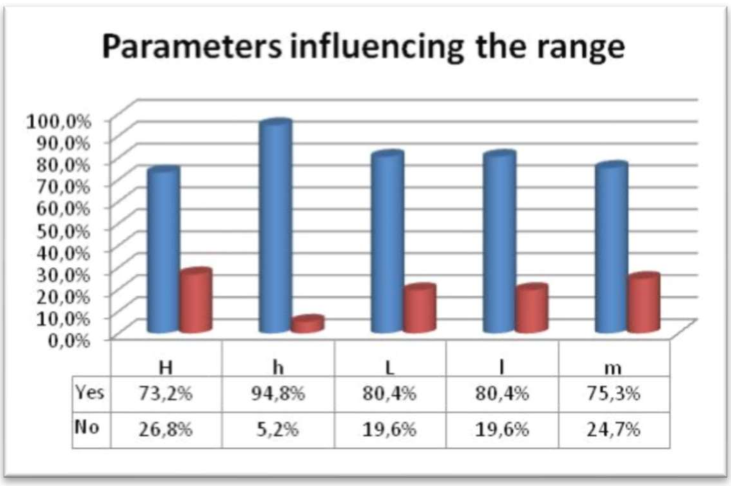


The measurements obtained provide a graphical representation that it is possible, knowing the usual functions of the mathematics course, to associate with one of the functions. It is normally clear from the objectives of the module that it is not a relationship of proportionality.

Due to the questions raised by this laboratory, we wondered about the student achievement at the end of the humanities, the ballistic fire being the subject of courses in fifth secondary (pupils of 17 years). In September 2012, we proposed a situation-problem to the students of the university preparatory course in physics at UCL. This situation-problem aims to allow us to analyze the reasoning of future students facing the inclined plane laboratory. Will they become aware of the influence of the instantaneous speed at the edge of the table and the time of fall to correctly answer the problem? This laboratory can also be treated in terms of parabolic trajectories from the edge of the table as Mersenne did in the context of the study of the water jet, see next paragraph on the parallelism between past and present.

Before experiments were conducted, students (97) were asked to indicate for each parameter of the drawing whether or not they influenced the range. They also had to justify their response and, in the case of influence, indicate in what sense. Finally, experiments were conducted to compare the observations with the initial responses to find an exact justification. Here are the results showing a strong majority in favor of the influence of the five parameters, but two parameters have an influence in the laboratory conditions.





An epistemological study shed light on the difficulties. We compare past research with that of the present on problem modeling and physico-mathematical physics.

Interested in the fall of bass, Galileo has implemented a modeling allowing simpler and easier measurements to analyze the free fall. He uses the inclined plane. He will thus establish the first fruits of the law of inertia.

In addition, Galileo studied horizontal shot as written in the *Discorso* according to Lindemann.

> I imagine that a mobile has been launched on the horizontal plane from which any obstacle has been removed; [...] the mobile I imagine endowed with a certain gravity, reached the end of the plane, continuing its course, will add to its previous uniform and indelible motion the downward tendency conferred on it by its gravity: the result will be this motion composed of a uniform horizontal motion and a naturally accelerated motion downwards. (Lindemann 2000, 130)

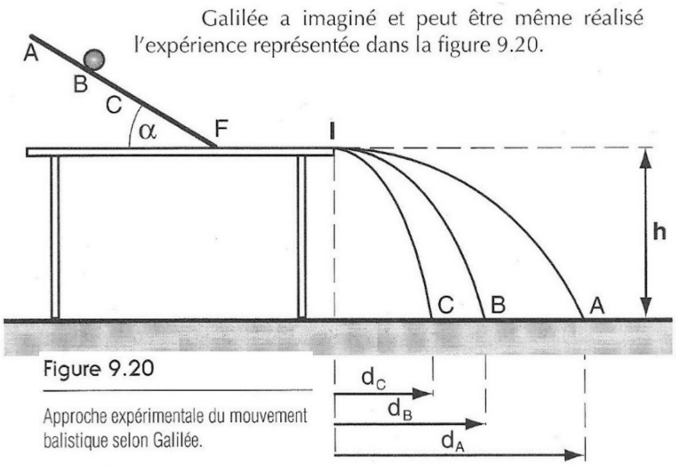


Fig. 2 Representation of a horizontal shot studied by Galileo in Lindemann (2000, 130).

Can one go so far as to assume or assert that Galileo realized the problem situation of the inclined plane followed by horizontal shooting as shown in the figure proposed by Lindemann?

Our epistemological analysis will aim to verify the existence in the past of the study of this problem-situation, to search for others, to compare past difficulties with those of the present; finally, to seek a new light from the mathematical knowledge acquired over the centuries that has led to advances in the understanding of physical phenomena.

It is obvious that this problem was studied in the seventeenth century and not only by Galileo. In the *Opere di Galileo Galilei* (1898) are handwritten notes related to horizontal shooting, including graphic representations:

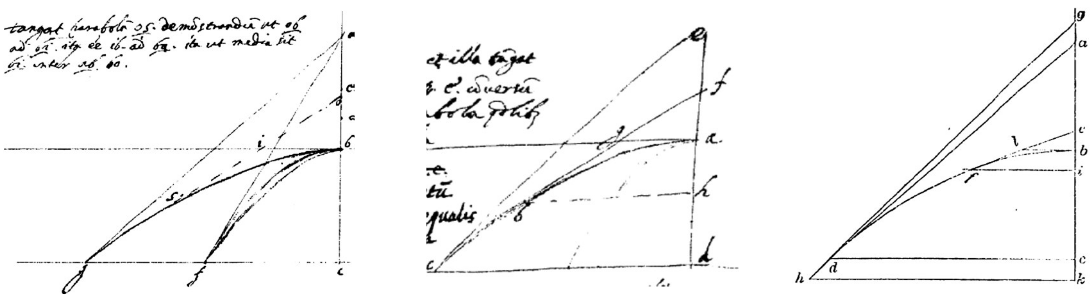


Fig. 3 Representation of a horizontal shot, 1898, *Le opere di Galileo Galilei.*

Similarly, in *Cogitata physico-mathematica, in quibus tam naturae quam artis effected admirandi certissimis demonstrationibus explicantur* (1644), Marin Mersenne (1588-1648) studied the jets of water. He draws interesting graphical representations (Fig. 4). Our interest in Mersenne is already in his title and his use of physico-mathematical, but it does not stop there. Indeed, it is interesting to note that it decomposes the problem, as it is done in the current teaching, by considering two cases, the horizontal shooting and the oblique shooting. It provides a superb representation with its technicality encompassing both situations.

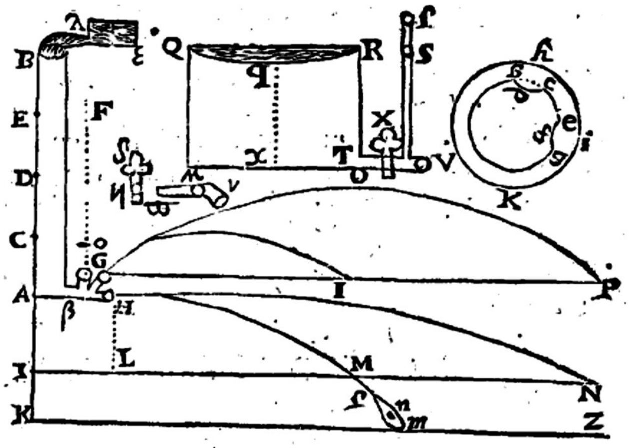


Fig. 4 Representation of the trajectory of water jets under different conditions.

This association of the two phenomena is remarkable and is in correspondence with the laboratory explained above where we observe a variation of the range as a function of the height of fall. Therefore, the two following figures of Mersenne give full meaning to physico-mathematics by associating height and parabolic trajectory (Fig. 5 and Fig. 6).

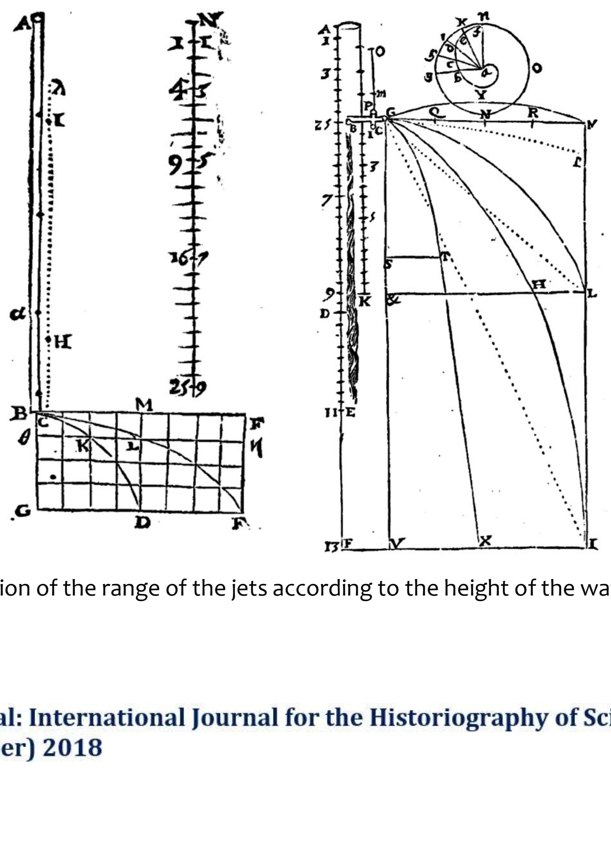


Fig. 5 Variation of the range of the jets according to the height of the water column.

Mersenne performs the same representations in the context of ballistic shot, thus providing a mathematical analogy, to read on the subject the article by Jean Dhombres, *De Marin Mersenne à Joseph Fourier : la boîte à outils graphiques du physico-mathématicien,* in Sciences et Techniques en Perspectives, 2016.

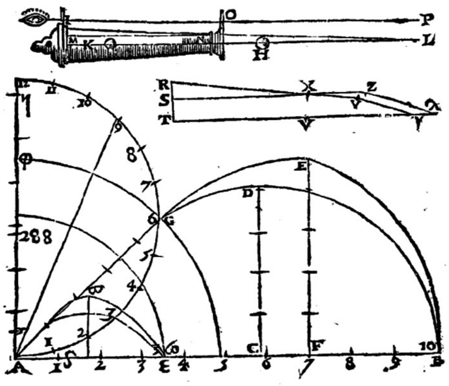


Fig. 6 Representation of the trajectory of ballistic shot under different conditions.

b) From Derivative to Differential and Speed to Acceleration: A World of Misunderstandings

Since the discovery of differential calculus, two views of the concept of derivatives have developed, one centered on ratios and inspired by Leibniz, the Bernoullis, l'Hôpital, Varignon and Euler, the other centered on functions, likewise inspired by Leibniz but also by Lagrange, Cauchy and others.

This development generated two different streams that must not be mixed, since otherwise misunderstandings among students will result, as already pointed out by Kac and Randolph:

> Granted, then, that we are not going to dispense with differentials, can we not do something to help students understand what they are? The unsophisticated undergraduate who tries to believe everything he reads, and his instructor tells him is hopelessly confused. One day he tries to believe (but does not succeed) that dy/dx is not dy divided by dx. The next day he may have momentary comfort when he learns that it is true after all that dy/dx is dy divided by dx and that $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ (Kac and Randolph 1942, 110)

As has been pointed out, this problem dates to the discovery of differential calculus.

In 1675 appeared the differential notation (Figure 7). Leibniz then questioned the possible equality between the differential of the product and the product of the differentials. It was the same for the quotients.

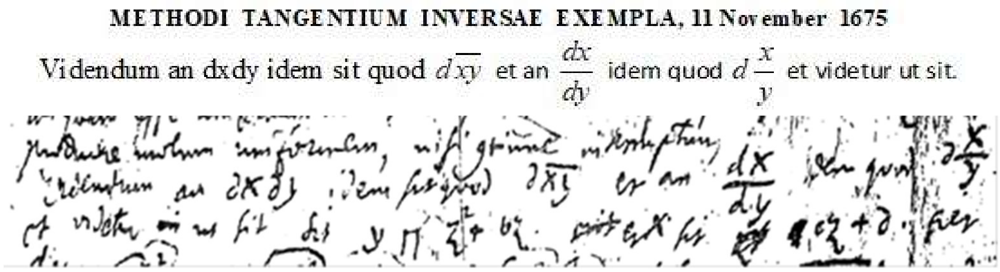


Fig. 7 Differential Quotient of Leibniz.

In 1713, during the quarrel of priority that pitted him against Newton, Leibniz wrote *Historia et Origo Calculi Differentialis*.[2] Taking note of the explanations of the graph (figure 8), we note that Leibniz indicates the character not necessarily infinitesimal of his computation:

> Our young friend, stimulated by this and pondering on the fertility of this point of view, since previously he had considered infinitely small things such as the intervals between the ordinates in the method of Cavalieri and such only, studied the triangle ${}_1YD_2Y$, which he called the Characteristic Triangle, [...] Even though this triangle is indefinite (being infinitely small), yet he perceived that it was always possible to find definite triangles similar to it. (Child 1920, 38-39)

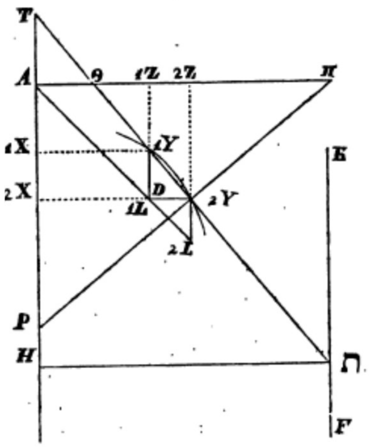


Fig. 8 Characteristic triangle.

Let us note that the differential quotient is indeed a ratio, that of the portions of coordinates and coabscisses.

This method[3] based on similarity ratios is still used by physicists to demonstrate formulas such as centripetal acceleration.

These reports were not necessarily infinitesimal as Marc Parmentier points out: "[…] for example the ratio between two differential quantities of the same order, which is no longer infinitesimal and […]" (Parmentier 1989, 38).[4]

[2] Translate from the Latin texts published by Gerhardt.
[3] Presented by M. Roland in October 2014 in Brussels during a study day of the ARCD.
[4] "[…] par exemple le rapport entre deux quantités différentielles de même ordre, qui n'est plus infinitésimal et, […] (Parmentier 1989, 38).

Leibniz, who defined his differential quotient as a ratio of quantities based on ratios of similitude in triangles, had doubts about the relevance of this, due perhaps to his research on the concept of functions. He mentioned this in a letter to Johan Bernoulli in 1698. Here is a translated excerpt: "I, too, often use differentiated functions, having abandoned the differentials, so that if z is the function of x itself, for me ~~d~~z is the ordinary quantity that is produced by dividing dz by dx, thus: ~~d~~z = dz:dx" (Leibniz 1698, 859).

Here he raised the central question of the meaning of quotient: a true ratio or a differentiated function. At this stage, the concept of a limit was not yet being used, because it had not yet been defined.

Let us consider how Benson (2009) deduced uniformly accelerated rectilinear motion (UARM) equations in his book, which is used in a number of first-year university courses in physics in French-speaking Belgium (Fig. 9).

$$\mathrm{v}_{inst} = \mathrm{v}_x(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$
$$dx = \mathrm{v}_x . dt$$
$$x_f - x_i = \int_{t_i}^{t_f} \mathrm{v}_x \, dt$$

$$a_{inst} = a_x(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathrm{v}_x}{\Delta t} = \frac{d\,\mathrm{v}_x}{dt}$$
$$d\,\mathrm{v}_x = a_x . dt$$
$$\mathrm{v}_{x_f} - \mathrm{v}_{x_i} = \int_{t_i}^{t_f} a_x \, dt$$

Fig. 9 Deduction of UARM equations, Benson.

We have come up against the epistemological obstacle described by Kac and Randolph. The use of the limit does not permit the transition from the first line to the second, since the derivative is not a quotient of differentials. Physicists, however, interpret it as a ratio, justifying this on empirical grounds relating to infinitesimal measurements. In his lectures, Richard Feynman, who won the Nobel Prize for Physics in 1965, refers to the notation used in the second line:

> Physicists like to write it *ds* = *v dt*, because by *dt* they mean Δ*t* in circumstances in which it is very small; with this understanding, the expression is valid to a close approximation. […] The quantity *ds*/*dt* which we found above is called "the derivative of *s* with respect to *t*" (this language helps to keep track of what was changed), and the complicated process of finding it is called finding a derivative, or differentiating. The *ds*'s and *dt*'s which appear separately are called *differentials*. (Feynman 2014, 117)

It is worth noting that Feynman uses the term quantity. Historically, he places himself within the framework defined mainly by Varignon (1698, 229), which we call **algebraic**:

> General rule. Velocities, Times, Spaces. $y = \frac{dx}{dz}$, or $dz = \frac{dx}{y}$, or $dx = y\,dz$

This is a ratio that is intentionally non-dimensional, to abstract it from the physical meaning of quantities, in accordance with Euclid's theory. Varignon says nothing about respecting homogeneous ratios in his works of 1698 and 1700, but he does mention it in 1707. Here is the excerpt:

> It should be pointed out here that since space and time are heterogeneous quantities, it is not specifically these quantities that are compared with each other in the ratio that is called velocity, but only the homogeneous quantities that express them, which are

> here and will always be in what follows either two lines, or two numbers, or any two other homogeneous quantities that are desired. (Varignon 1707, 223)

Lévy-Leblond indicates how difficult it was for Galileo to work with the existing theoretical framework:

> It could be said that modern physics dates back to Galileo and the considerable efforts that he had to make over a number of years, almost decades, to explain for himself the notion of instantaneous velocity. Looking at the long intellectual struggle in which he engaged to understand what an instantaneous velocity is, to understand what a constantly changing velocity is (clearly he did not yet have differential calculus at his disposal and was obliged to work on this question using Euclid's theory of proportions, which is not very convenient), one realizes that it was precisely this capacity to use mathematical conceptualization that allowed a physical concept to emerge, in an embryonic form, which would not be developed to the full until later with the appearance of differential calculus and the notion of derivatives. (Lévy-Leblond 1991, 35)

This excerpt shows that it was necessary to combat the existing theories. In this context, Gaston Bachelard talks about epistemological obstacles: "When the mind addresses scientific culture, it is never young. Indeed, it is very old, because it is the same age as its prejudices. To attain to science is to be spiritually rejuvenated, to accept a sudden change that contradicts the past" (Bachelard 1993, 14).

The aim here is not to misrepresent mathematical conceptualization. The use of a limit dates back to the framework defined by Cauchy, which we call **analytical**:

> The same will apply in general; but the form of the new function that will serve as a limit to the ratio $\frac{f(x+i)-f(x)}{i}$ will depend on the form of the proposed function y = f(x). To indicate this dependency, the new function is called a *derivative function*, and it is designated, using an accent, by the notation y' or f'(x). (Cauchy 1823, 9)

Following Varignon, Euler continues in this line of the algebraic framework as evidenced by excerpts from Theoria *Motus Corporum Solidorum Seu Rigidorum* Vol.1, translated into English by Ian Bruce.

Theoria motus corporum solidorum seu rigidorum

Euler, Leonhard

Rostochium et Gryphiswaldiae, 1790

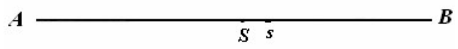


Fig. 2

PROBLEMA. 1.

42. Si punctum in linea recta moveatur, univerſam motus determinationem ad calculum revocare.

SOLUTIO.

Totum negotium huc redit, ut ad quodvis tempus locus aſſignetur, ubi tum punctum reperiatur. Sit ergo AB linea recta, in qua punctum incedat, initio in A conſtituto, atque elapſo tempore = t, pervenerit in S, ſtatuaturque AS = s, quod erit ipſum ſpatium tempore *t* deſcriptum. Quodſi jam inter *t* et *s* aequatio detur, qua alterum ex altero definiri queat, inde omnia, quae ad motus cognitionem pertinent, innoteſcent. Differentiatione enim inſtituta pro temporis elemento *dt* ſpatii elementum *ds*, quod eo percurritur, derivatur: atque fractio $\frac{ds}{dt}$ celeritatem puncti in S exprimet. Conſtat enim, hanc fractionem continere quantitatem finitam. Quare ſi celeritas in S ponatur = *v*, erit $\frac{ds}{dt} = v$, unde tam ad quodvis tempus, quam ad quemvis ſpatii locum, celeritas aſſignari poterit. Directio autem motus ubique cum ipſa recta AB congruet.

COROLL. 1.

43. Si ad ſingula temporis momenta celeritas corporis detur *v*, ita ut relatio inter *t* et *v* conſtet, inde quoque ſpatia *s* ſingulis temporibus *t* deſcripta definientur ope aequationis $ds = v\,dt$, cujus integrale praebebit ipſum ſpatium $s = \int v\,dt$.

PROBLEM 1

42. If the point is moving in a straight line, the whole determination of the motion can be referred to calculation.

SOLUTION

The whole work is reduced to this, that for whatever time the place has been assigned, then the point will be found there. Hence let *AB* be a right line (Fig. 2), on which the point lies, initially constituted at *A*, and with the elapse of a time equal to *t* it arrives at *S*, and there is put in place *AS* = *s*, which is the interval described in the time *t*. But if now the equation is given between *t* and *s*, from which the one from the other is able to be defined, thus everything can become know relating to the motion. For by differentiation with the element of time *dt* put in place, the element of distance *ds*, which is traversed in that time, is derived, and the speed of the point at S is expressed by the fraction $\frac{ds}{dt}$. For it is agreed that this fraction retains a finite quantity. Whereby, that speed at S is put equal to *v*, and it becomes $\frac{ds}{dt} = v$, thus the speed can be assigned for whatever the time and position of the interval. Moreover the direction of the motion agrees everywhere with the line *AB*.

COROLLARY 1

If the speed of the body *v* is given at individual moments of time, thus in order that the relation between t and *v* agrees, then the distances *s* described at individual times *t* can be defined with the help of the equation *ds* = *vdt*, the integral of which gives the distance *s* = ∫*vdt*.

48. Ut locus curvae S per coordinatas OX = *x* et XY = *y* determinatur, ita locus ſequens *s* per earum elementa *dx* et *dy* definitur ſcilicet punctum ex S egreſſum tempuſculo *dt* ſecundum directionem OA per ſpatiolum *dx*, ſecundum directionem O B vero per ſpatiolum *dy* transfertur.

48. As the position of *S* on the curve is determined by the coordinates *OX* = *x* and *XY* = *y* , thus the position of the following point *s* is defined by the elements of this *dx* and *dy*; clearly the point departing from S in the small time *dt* is transferred along the direction *OA* by the small distance *dx*, and now along the direction *OB* by the small distance *dy*.

In Euler's text, we find the three forms of writing Varignon's text as well as the method of separating variables for integration.

You also find in both the reports but also the consideration of "infinitely" small elements, from Leibniz.

It is Lagrange who combines the concept of function with that of derivative and instantaneous speed, thus creating an analytical framework. Lagrange expresses reservations about the rigor of the work of some of his predecessors. He continues the work of Leibniz and Euler by deepening the concept of function and sets up a functional theory overcoming the shortcomings of the new Calculus. In his work of 1797 *Théorie des fonctions analytiques contenant les principes du calcul différentiel dégagés de toute considération d'infiniment petits et d'évanouissans, de limites ou de fluxions et réduits à l'analyse algébrique des quantités finies*, the title clearly evokes the break with the theories of the past, it defines the functions that derive from a given function.

> This way of deducing, from a given function, other derived functions, and depending essentially on the primitive function, has the greatest importance in analysis. The formation and the computation of these various functions are, strictly speaking, the true object of the new computations, that is to say of the so-called differential, or fluxional calculus. The first geometers who have used the differential calculus, namely Leibniz, Bernoulli, l'Hospital [...] have based it on the consideration of infinitely small quantities of different orders, and on the supposition that we can look at and consider as equal the quantities which only differ from each other by infinitely small quantities with respect to them. Happy to arrive by the presses of this computation in a prompt and safeway to exact results, they did not care themselves with proving the principles. Those who followed them, Euler, d'Alembert [...] have tried to overcome this defect, by showing, through particular applications, that the differences which are supposed to be infinitely small must be absolutely equal to zero; and that their ratios, the only quantities which really enter into the calculus, are nothing but the limits of the ratios of finite or indefinite differences. (Lagrange 1797, 2-3)

> One knows the difficulties offered by the supposition of the infinitely small, on which Leibniz founded the differential calculus. To avoid them, Euler considers the differentials as equal to zero, which reduces their ratio to the expression zero divided by zero, which means nothing. [...] Maclaurin and D'Alembert use the consideration of limits and consider the ratio of the differentials as the limit of the ratio of the finite differences, when those differences become zero. This way of representing the differential quantities only postpones the difficulty; because, in the final analysis, the ratio of the vanishing differences is again reduced to that of zero to zero. Moreover, it can be observed that the known word of limit is improperly applied to what becomes an analytic expression when some quantities are made equal to zero, because these limits, after decreasing to zero, could still become negative. In the same way as in geometry, one cannot say that the sub-tangent is the limit of the sub-secants, because nothing prevents the sub-secant from growing further when it has become a sub-tangent. [...] The true limits, according to the notions of the ancients, are quantities which one cannot pass, although one can approach them as near as one wishes; it is the case, for example, of the circumference of the circle with respect to inscribed and circumscribed polygons, because no matter how large the number of sides becomes, neither the inner polygon will ever leave the circle, nor the exterior will enter it. Thus, the asymptotes are true limits of the curves to which they belong, etc. (Lagrange 1807, 1-2)

Cauchy, meanwhile, estimated in 1822 that, during the development of his theory and notation (1797), Lagrange did not show a great rigor:

> It is therefore not allowed to substitute indistinctly the series to the functions, and to be sure of making no error, one must restrain this substitution to the case where the functions, being expandable in convergent series, are equivalent to the sums of these series. (Cauchy 1822, 51)

Let us add that Lagrange devoted a paragraph of his theory of functions (1797) to mechanics and more precisely to kinematics with the associated equations.

> So, in general, in any rectilinear motion in which the corresponding space is a given function of the elapsed time, the prime function of this function will represent the velocity, and the second function will represent the accelerating force at any instant; for, as the times, spaces, velocities, and forces are heterogeneous things, which can only be compared together after reducing them to numbers by relating each of them to a fixed unity of its specy, we can, for the sake of simplicity, express immediately the speed and the force by the prime and second functions, as we express the space by the primitive function. From that we see that the prime and second functions naturally occur in mechanics, where they have a definite value and meaning: this is what led Newton to base the calculus of fluxions on the consideration of motion. In this way, the space, velocity, and force being considered as functions of time, are respectively represented by the primitive function, its prime function, and its second function; so that, knowing the expression of space with respect to time, we will immediately have those of speed and force by a direct analysis of functions; but if we only know the speed or the force with respect to time, we must then go back to the primitive equations by the rules of inverse analysis. (Lagrange 1797, 226)

The analytical framework lacked rigor and the latter only arrived with the advent of the infinitesimal calculus set up by Cauchy. We find in *Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal*, lessons which deal with the limits, the infinitely small, the derivatives and finally the differentials of the functions of a single variable.

Based on this epistemological research, we thus note a dangerous mixing of the two frameworks by Benson, as described by R. Courant and H. Robbins:

> Before passing to the limit, the denominator Δx in the quotient Δy/Δx is cancelled out or transformed in such a way that the limiting process can be completed smoothly. This is always the crucial point in the actual process of differentiation. Had we tried to pass to the limit without such a previous reduction, we should have obtained the meaningless relation Δy/Δx = 0/0, in which we are not at all interested. Mystery and confusion only enter if we follow Leibniz and many of his successors by saying something like this: "Δx does not approach zero. Instead the *last value* of Δx is not zero but an *infinitely small quantity*, a *differential* called *dx*; and similarly, Δy has a *last* infinitely small value *dy*. The actual quotient of these infinitely small differentials is again an ordinary number […]". (Courant and Robbins 2015, 481)

To preserve the idea of the ratio, a number of works use empirical proofs ("it is no longer possible to measure below a certain value") that tend to impair students' understanding. It is possible, however, to keep intact the idea of a ratio, as explained by Kac and Randolph. To do this, it is necessary to introduce the concept of differentials and not to define the quotient as a limit. At what point should it be introduced? How is it possible to ensure reciprocity of the objects of knowledge between the worlds of mathematics and physics? Who will take responsibility for introducing the concept that allows others to use that object as a tool within their own discipline? Here we enter into the joint action theory of Gérard Sensevy (2011).

Another way of keeping the report is the use of non-standard analysis, which appeared in the sixties. However, this remedy does not solve the problem of teaching denounced by

Kac and Randolf, but also by Poincaré (1904) and Hadamard (1923). Indeed, in teaching, the predominant predominance is given to the standard analysis and the non-standard analysis comes much later in the university course. This justification can therefore only be used in secondary education and probably during the first academic years. Moreover, this recourse gives the impression of simplicity of easy justification by simply citing a theory more complex than it seems. It is therefore preferable not to use it at the beginning of learning, but rather as an improvement.

What is the value of keeping the differential quotient as a true ratio? What does it add? It is not a technical method that causes misunderstandings, and it is possible to do without it.

Is it necessary to use the differential quotient to deduce UARM equations? Do we really need the derivative for this deduction?

Merton's rule (Fig. 10), current formalism: $v_m = \frac{v_f + v_i}{2}$

The answer to the previous two questions is no. Merton's rule is sufficient to deduce the equations. The derivative is only useful once you begin studying harmonic movements.

Likewise, how can the transition from the second line to the third be proven (Fig. 9)? The notation x'(t)=v(t) makes it possible to integrate both sides of the equality in respect of t but what should you do in cases where dx = v dt? Is this an integration based on different bounds (in x and in t)? And what slices are used? What is the definition of the integral?

Finally, are the difficulties raised by Poincaré (1904) not important, and would it not be better to avoid introducing the concept of differentials too early?

In conclusion, this epistemological duality shows that there has been a dual approach since the discovery of differential calculus. This dual approach offers definite benefits. Mixing the approaches, however, creates an epistemological obstacle to understanding for many students. The only way of transcending the difficulty of the indeterminate 0/0 (which is a difficulty for a large number of students) is to use the limit, which makes it impossible to utilize the differential quotient as a true ratio. Empirical proofs do not remove this ambiguity and give rise to errors of interpretation and definition.

Its usefulness is in distinguishing the approaches that we find in the solutions to problems.

Applied Duality

Let us consider solving kinematic problems in order to pinpoint possible changes of framework within solution methods. Do these methods reveal a more mathematical, analytical approach, or a more physics-based, algebraic approach?

We submitted three kinematics problems to 817 first-year undergraduates (physics, mathematics, chemistry, business engineering and civil engineering), which yielded some surprisingly successful results: 54.2% for problems of the first type, 35.5% for the second and 8.3% for the third. Due to the low response rate (96 out of 817), the third problem was presented at the qualifying stage of the physics Olympiad, with a success rate of 11.7% in the fifth year of secondary school (275 students aged approximately 17 years) and 22.7% in the sixth form (233 students aged approximately 18 years). A similar exercise was then given to fifth-year students who were finalists in the tests (62 students) for a detailed solution, with a success rate of 38.7%.

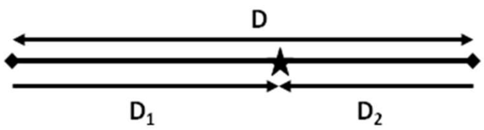
A number of students also stated that they had relied on their university physics courses and their practical work to solve the exercises.

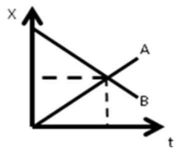
For fuller explanations on this subject, we invite the reader to see our article in the collective work *Épistémologie et didactique* (2017), or the associated poster presentation (2014). In this article, we will briefly consider the first and third problems.

The first problem is set out as follows:

> Two slugs start moving towards each other at 5 am. The distance between them is 8 m. The first covers 50 cm in half an hour and the second moves 20 cm in 20 minutes. At what time will they meet and at what distance from the first slug's point of departure?

Below is a table summarizing the methods used by the students. It is divided into two frameworks, each of which is divided into two conceptual windows (CWs).

<table>
<tr><th>ALGEBRAIC FRAMEWORK</th><th>ANALYTICAL FRAMEWORK</th></tr>
<tr><td>CW: algebraic relationship</td><td>CW: numerical relationship</td></tr>
<tr><td>

$D_1 + D_2 = D \Leftrightarrow v_A.t + v_B.t = D$
$\Leftrightarrow t = \frac{D}{v_A + v_B}$</td>
<td>

<table>
<tr><th>time</th><th>Position A</th><th>Position B</th></tr>
<tr><td>t_1</td><td>$x_A(t_1)$</td><td>$x_B(t_1)$</td></tr>
<tr><td>t_2</td><td>$x_A(t_2)$</td><td>$x_B(t_2)$</td></tr>
<tr><td>t_3</td><td>$x_A(t_3)$</td><td>$x_B(t_3)$</td></tr>
<tr><td>t_4</td><td>$x_A(t_4)$</td><td>$x_B(t_4)$</td></tr>
<tr><td>t_5</td><td>$x_A(t_5)$</td><td>$x_B(t_5)$</td></tr>
</table></td></tr>
<tr><td>CW: relative velocity</td><td>CW: numerical function</td></tr>
<tr><td>Relative velocity: $v_B + v_A$
Distance: D
Algebraic ratio: $t = \frac{D}{V_A + V_B}$</td>
<td>$\left.\begin{array}{r} x_A = v_A.t \\ x_B = x_0 - v_B.t \end{array}\right\} x_A(t) = x_B(t)$
$\Leftrightarrow v_A.t = x_0 - v_B.t \Leftrightarrow t = \frac{x_0}{v_A + v_B}$</td></tr>
</table>

In this summary we note the duality of approaches, which is similar to what existed when differential calculus was discovered. The algebraic framework, which is closer to the mindset of physicists, is based on looking for a ratio between quantities, while the analytical framework, which is closer to the mindset of mathematicians, is based on the concept of functions as understood at different stages. The students have encountered these methods during their studies. They were explained by teachers with a university education. It is important to keep these multiple approaches, since they allow each student to adopt the method that he or she finds suitable and understandable, as well as building bridges between the two scientific disciplines.

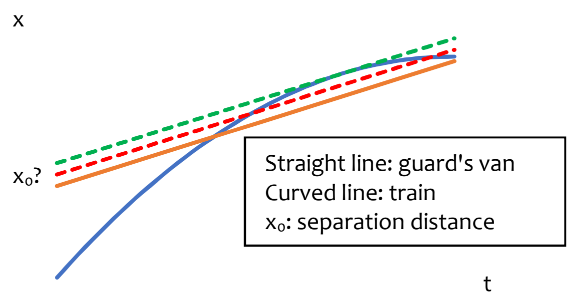
As Lévy-Leblond writes, physicists and mathematicians are trained differently but it is important for them to be able to work together:

> We do not do the same job, and it is normal, since we have different ways of doing things, that we should have different ways of saying or writing things. I think this encounter is very enriching. That is what gives science its vitality, whether within a

discipline or at the interface between different disciplines, it is not their convergence towards becoming unified and tending towards a single formulation or towards identical statements; on the contrary it is their diversity. (Lévy-Leblond, 1991, 45)

The next problem caused greater difficulties. Nevertheless, it is an interesting example for use in the context of interdisciplinarity between physics and mathematics.

The driver of a train moving at 100 km/h notices, on the same track 80 m away, the guard's van of a train moving in the same direction as him at 28 km/h. He applies the brakes, which results in deceleration of 2 m/s². Will there be a collision? At what minimum distance must the guard's van be noticed in order to avoid a collision?

ALGEBRAIC FRAMEWORK	**ANALYTICAL FRAMEWORK**
CW: algebraic relationship	**CW: numerical relationship**
No methods of this type for this problem, but see the analysis put forward in our doctoral research on problems involving objects moving towards each other or in the same direction at different velocities in uniform rectilinear motion (URM).	
CW: relative velocity	**CW: numerical function**
Relative velocity: $\Delta v = v_B - v_A$ Time (relative velocity zero): $t = \frac{\Delta v}{a}$ algebraic ratio Merton's Rule: $v_m = \frac{v_f + v_i}{2}$ Minimum distance: $x_0 = t \,.\, v_m$	$\left.\begin{array}{l} x_A = V_A.t - \frac{at^2}{2} \\ x_B = x_0 + V_B.t \end{array}\right\} x_A(t) = x_B(t)$ $\quad V_A.t - \frac{at^2}{2} = x_0 + V_B.t \Leftrightarrow \frac{at^2}{2} - (V_A - V_B).t + x_0 = 0$ $\Delta = (V_A - V_B)^2 - 2.a.x_0 \Leftrightarrow t = \frac{(V_A - V_B) \pm \sqrt{\Delta}}{a} \Leftrightarrow x(t) = \ldots$ $\Delta = 0 \Leftrightarrow x_0 = \frac{(V_A - V_B)^2}{2a}$

An analysis of the mistakes made by the students and by the 62 finalists (38.7% were successful) when tackling a similar problem in the Olympiads will be set out in our thesis.

Here is the table summarizing the methods used by the students.

This example shows the advantage of graphical representation. It gives teachers a way of correcting an error committed by a large number of students, who tend to calculate the time taken for the train to stop and then look for the position of the train and the guard's van at that time. They compare the positions and then reach a conclusion on whether or not there is a collision. On the graph, however, the middle secant line would suggest the incorrect conclusion that they do not collide, since the position of the guard's van is further away than that of the train. It is therefore useful to combine the methods. Few students instinctively use an unknown parameter to represent the initial position. The algebraic method then makes it possible to overcome this obstacle.

Moreover, the existence of a graphical solution is proven thanks to Lagrange's finite-increments theorem, which is an example of interdisciplinarity. This theorem indicates that finding the minimal distance comes down to looking for the point in time when the velocity of the two trains is the same, which is a change of framework.

This shows the value of juggling different methods depending on the situation and making use of a change of framework, seeking either the minimum distance or the time when the guard's van and the train have the same velocities.

Theoretical Duality

We will now analyze another example: the deduction of the formula for centripetal acceleration.

Let this be a particle rotating at any velocity about a point at a distance R (Fig. 11). We find the position of the point on the circle at every instant, then its velocity by the derivative (using the limit) and finally its acceleration by the second derivative.

$$\begin{cases} x(t) = R\cos\theta(t) \\ y(t) = R\sin\theta(t) \end{cases} \quad \begin{cases} \dot{x} = -R\sin\theta\,\dot{\theta} \\ \dot{y} = R\cos\theta\,\dot{\theta} \end{cases}$$

$$\begin{cases} \ddot{x} = -R\cos\theta\,\dot{\theta}^2 - R\sin\theta\,\ddot{\theta} \\ \ddot{y} = -R\sin\theta\,\dot{\theta}^2 + R\cos\theta\,\ddot{\theta} \end{cases} \quad \vec{a} = R\dot{\theta}^2\,\vec{n} + R\ddot{\theta}\,\vec{t}$$

By defining the radian: $s = R\theta \quad \vec{a} = \dfrac{\dot{s}^2}{R}\vec{n} + \ddot{s}\,\vec{t}$

We deduce from the general formula the particular case of uniform circular motion: $a = \dfrac{v^2}{R}$

So this is a ***deductive inference*** (Barth, 1987).

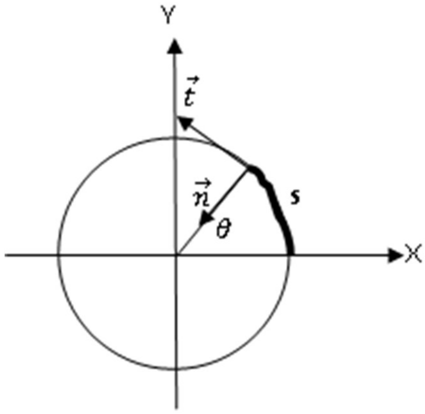


Fig. 11 Circular motion.

In the excerpt from Benson cited below, the author sets out from the particular case of uniform circular motion, for example an object rotating on a turntable, which is an observation:

> The figure (Fig. 11 bis) represents a particle moving at a constant velocity modulus *v* along a circle with radius *r*. This is **uniform circular motion**. Suppose that, during a short time interval Δ*t* its position vector turns through angle Δϑ, and the displacement of the particle is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$, i.e. vertical. Since $\vec{v}$ is always perpendicular to $\vec{r}$, the directions of these two vectors vary by the same angle during any time interval. On the vector diagram for the equation $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$, we note that $v_2 = v_1 = v$. The direction $\Delta\vec{v}$ is horizontal and radial towards the center, and overlies the bisector of the angle Δϑ inside the circle. The triangles *OPQ* and *ABC* are two isosceles triangles with identical angles. (Why?) Therefore, $\dfrac{|\Delta\vec{r}|}{r} = \dfrac{|\Delta\vec{v}|}{v}$ and from this we get $|\Delta\vec{v}| = (\frac{v}{r})|\Delta\vec{r}|$. Since $|\Delta\vec{r}| \approx v\Delta t$

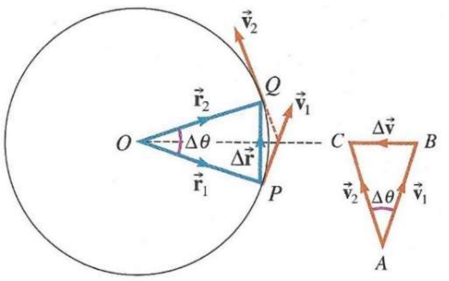


Fig. 11 bis Uniform Circular motion, Benson.

we see that $\frac{|\Delta\vec{v}|}{\Delta t} \approx \frac{v^2}{r}$. According to the definition $\vec{a} = \lim_{\Delta t \to 0} (\frac{\Delta \vec{v}}{\Delta t})$, we find that the centripetal acceleration modulus is $a_r = \frac{v^2}{r}$.

In this approach, the author uses the ratio of similitude between two similar triangles, as Leibniz did on the basis of his characteristic triangle. This is a non-dimensional ratio between moduli. He then generalizes by summing the centripetal and tangential accelerations ($a_t = \frac{dv}{dt}$), thus obtaining the resulting acceleration for non-uniform circular motion. He should then have validated this conjecture. This is therefore an ***inductive inference*** (Barth, 1987).

Comparing the two approaches, we observe that the inductive inference uses an arbitrary process of thought, a generalization, which is bold and requires *a posteriori* validation, which is often forgotten. This is an example of an obstacle to general knowledge described by G. Bachelard:

> We will have a much easier battle when we are able to show that seeking to generalize too hastily very often leads to misplaced general statements, which are disconnected from the essential mathematical functions involved in the phenomenon. (Bachelard 1993, 56)

In the deductive inference, the use of derivatives, passing to the limit and avoiding the manipulation of "infinitely small" quantities brings us back to the **analytical framework**.

In the inductive inference, the use of the ratio of similitude and of approximations ("short time interval", which is infinitely small) before passing to the limit refers back to the **algebraic framework** in which the ratio between quantities is used, the Δs, as put forward by Feynman (2014).

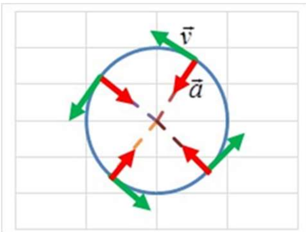
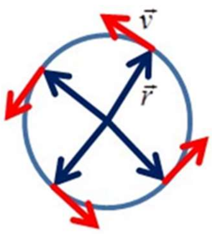
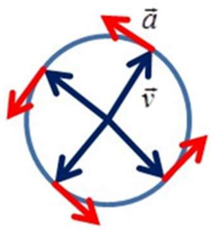
Uniform circular motion (UCM)	
Decomposition and analogy	
Position vector revolution	Speed vector revolution
 $T = \frac{2\pi R}{v}$	 $T = \frac{2\pi v}{a}$

Fig. 12 Research of the centripetal acceleration formula.

To avoid the approximations of Benson's proof, it is possible to propose another approach to the centripetal acceleration formula (Fig. 12), but in the case of the UCM alone. This approach is inspired by a proposal used during his lectures by Philippe Léonard, director of the physics experimentarium of the Université Libre de Bruxelles.

Instantaneous speed: $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$. Instantaneous acceleration:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}.$$

For the uniform circular motion:

$$\|\vec{r}\| = R, \|\vec{v}\| = v \text{ et } \|\vec{a}\| = a.$$

By the equality of periods:

$$\frac{2\pi R}{v} = \frac{2\pi v}{a} \Leftrightarrow a = \frac{v^2}{R}.$$

We will now also show a dual approach that is observed in courses when justifying the following approximation sin(x) ≈ x for sufficiently small angles.

Mathematicians tend to explain this based on the limit of the ratio sin(x)/x tending towards 1 or graphically based on the representation of the two functions in the same Cartesian coordinate system (Fig. 13), where x is a tangent to the graph of sin(x) at 0. So, this involves the use of the concept of functions.

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

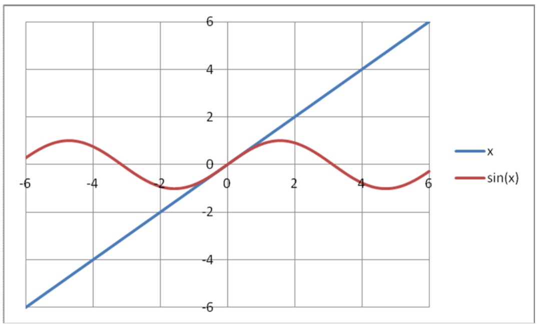


Fig. 13 Sin(x) and x

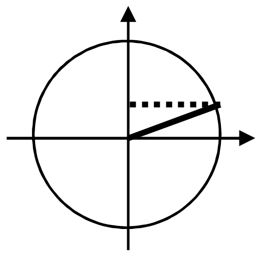


Fig. 14 Trigonometric circle

Physicists tend to explain it on the basis of the trigonometric circle (Fig. 14), by comparing the length of the arc (x) and its projection on the ordinate (sin(x)). This involves the use of a comparison of lengths or quantities.

This duality shows that there is a difference of approach between mathematicians and physicists, even though the underlying reality is the same. So, we come back to Lévy-Leblond on the importance of diversity.

Duality in Modelling

The deduction of UARM equations requires modelling using first-order differential equations with integration. The danger is in establishing a method for "separation of variables" that dissimulates the composition of functions or changes in variables, as pointed out by Poincaré (1904) and Hadamard (1923).

> It is therefore necessary to learn to think in terms of derivatives; once this habit has been acquired, Leibniz's notation can be safely used. It is clear that the best way to inculcate this habit in students is to teach them Lagrange's notation first. Once they have familiarized themselves with this language and have used it in a number of exercises, and they know how to do a change of variables, they can then be told about Leibniz's notation without any difficulties. [...] If, however, one seeks to teach them to do changes of variables using Leibniz's notation from the beginning, they will never be able to do them correctly. (Poincaré, 1899, 109)

We can illustrate this excerpt from Poincaré based on the deduction of UARM equations, without forgetting that, as mentioned previously, this process does not require the use of a derivative. Let us consider the case of acceleration in Fig 9.

Starting from the definition of constant acceleration, how can the velocity be obtained correctly, whatever the notation used? (Fig. 15)

$$v' = a$$
$$\Leftrightarrow \int_0^t v'(s)\,ds = \int_0^t a\,ds$$
$$\Leftrightarrow v(t)-v(0) = a(t-0)$$

$$\frac{dv}{dt} = a$$
$$\Leftrightarrow \int_0^t \frac{dv}{ds}\,ds = \int_0^t a\,ds$$
$$\Leftrightarrow v(t)-v(0) = a(t-0)$$

Fig. 15 Appropriate method for the finding of speed.

$$\frac{dv}{dt} = a$$
$$\Leftrightarrow dv = a\,dt$$
$$\Leftrightarrow v(t)-v(0) = \int_0^t a\,ds$$

$$\frac{dv}{dt} = a$$
$$\Leftrightarrow \int_0^t \frac{dv}{\cancel{ds}}\,\cancel{ds} = \int_0^t a\,ds$$
$$\Leftrightarrow v(t)-v(0) = \int_0^t a\,ds$$

Fig. 16 Method generating epistemological obstacle

The other method (Fig. 16), this is what gives rise to misunderstandings among students, and some teachers or authors are not necessarily aware of it.

In the left-hand column, the transition from line 1 to line 2 brings us back to the epistemological obstacle described by Kac and Randolph (1942). The transition from line 2 to line 3, on the other hand, is never proven. How can this be integrated and how can the equality be proven? It is worth noting that in Benson's work, the second line was added between the second and fourth editions, which shows that there was a comprehension problem. That has certainly not been remedied, however, by this addition.

In the right-hand column, this dangerous simplification is based on the idea of a quotient and on a particular significance that is given to d. But how can this then be integrated? There ought to be an explanation for the change of variables dv = v'dt, which brings us back to the previous correct framework.

Finally, does the proof come from an extension of Leibniz's finite differences, allowing only the extremes to be kept v(t)-v(0)?

This misrepresentation of the concept of a derivative is also found in works written by mathematicians. Let us consider the example of Jean-Pierre Demailly (1996).

In the paragraph on equations with separate variables, he uses the differential quotient as a ratio from the outset when it is in fact a derivative, and the differential is not defined. We do not find this method in courses on classical analysis in reference to differential equations.

Equations y' = g(y) with g: J → $\Re$ **continuous.**

The equation can be written $\frac{dy}{dx} = g(y)$ or $\frac{dy}{g(y)} = dx$ provided that g(y) ≠ 0.

The solutions are given by G(y) = x + λ. (where G is any primitive of 1/g)

What are the grounds to justify misrepresenting the concept of a derivative?

It would have been simpler and more correct to set out from the equation in terms of derivative y'=g(y) and then to write, subject to the condition that g(y)≠0, $\frac{y'}{g(y)} = 1$ and finally to integrate $\int \frac{y'}{g(y)} dx = \int 1\, dx$, which can be solved by knowing that $(G \circ y)'(x) = y'.\frac{1}{g(y)}$.

What is more surprising is that in the previous chapter, which sets out all the important theorems, there is an example in which the method at issue is not used. This involves the solution to the equation y'= y². The method is exactly as set out above.

This misrepresentation of the concept of a derivative by a mathematician does not take any account of the epistemological obstacle described by Kac and Randolph (1942) or the dangers highlighted by Poincaré and Hadamard.

To illustrate our statement, we will use a piece of work carried out in 1996 (Campion et al.) by a working group comprising three lecturers and three students. This group looked at the difficulties encountered by the students in the applied science faculty at UCL in reading handouts. The work was published under the title *Dialogues concernant deux sciences*. It is written paraphrasing Galileo in his *Dialogo sopra i due massimi sistemi del mondo, Tolemaico e Copernicano*. Nathanaël is a bright student who asks questions when put in front of some argumentations of physicists he does not understand. Here is an excerpt of a problem of misrepresentation giving rise to misunderstandings.

Nathanaël – To solve the differential equation $y'^2 = \frac{a^2 - y^2}{y^2}$ some people follow this method. They state that it is equivalent to $\frac{dy}{dx} = \pm\frac{\sqrt{a^2 - y^2}}{y}$ (1) and begin by considering equation (1) with the positive sign. The variables are separated, they say, so you get $\frac{y\,dy}{\sqrt{a^2 - y^2}} = dx$. Therefore $x = \int y(a^2 - y^2)^{-\frac{1}{2}} dy$ …

Nathanaël – This loss of meaning is completely obscured by the calculations that are carried out. In any case, I do not understand them properly. What meaning should be given to the equation $\frac{y\,dy}{\sqrt{a^2 - y^2}} = dx$?

They seem to be multiplying by *dx*, which is a mystery to me.

At Paris-Diderot University in the 1990s, the dangers of infinitely small quantities (written as dx) and their interpretation, which are often used in physics by engineers and physicists, were set out. Is it not problematic to work with these and sum them without any precautions?

We refer to this study, *Procédures différentielles dans les enseignements de mathématiques et de physique au niveau du premier cycle universitaire.* There we find an example of an error of analogy between seeking to find the volume of a sphere and its surface area by cutting it into infinitesimal cylindrical slices. There is a problem of approximation when passing to the limit, due to the Schwarz paradox for the surface area. It is important to verify that the chosen approximation is valid, since otherwise a mistake will be made when passing to the limit. The surface areas of infinitesimal cylinders do not tend towards the surface area of the sphere.

This problem, the Schwarz paradox, is also present when modelling the physical phenomenon using meshing. As explained in the case of the surface area of the cylinder, the result cannot be found from all the meshed triangles on the surface. All the continuation lines fail to converge towards the surface of the cylinder. It is interesting to note that this type of problem can be solved using a new form of algebra reworked by mathematician Paolo Roselli (2014). This would involve a possible collaboration between mathematicians, physicists and engineers. We invite the reader to find out about the interesting work done by this researcher.

As Poincaré points out, the difficulty of using differentials is more striking once you move on to second-order derivatives, whereupon you are plunged into absurdity.

The separate variables method cannot be used in the situation that arises in second-order ODEs. In fact, it is impossible to multiply directly by $(dx)^2$ because of the d^2y, which has no meaning. So how the problem of the simple pendulum and its associated ODE should be solved? During interviews carried out in the course of our research, we identified a communication problem between mathematicians and physicists at Mons University. The mathematician did not understand how his students had never heard about ODEs in their dynamics classes.

In a paper entitled *La transposition didactique et son "triangle": le pendule simple comme exemple* (seminar entitled *Fondements et notions fondamentales*, 2015), we presented an analysis of different approaches to the simple pendulum.

Harmonic motion is studied in the final year of secondary school (sixth form). Most textbooks rely on the idea of establishing a concordance between three types of motion (uniform circular motion [UCM], sinusoidal rectilinear motion [SRM] and simple harmonic motion [SHM]), as presented in a book by Hecht (1999), but also in The Feynman Lectures on Physics, Volume 1, Chapter 21, Fig. 21-3. It is possible to work on the interpretation of this situation with students using a spreadsheet (Fig. 17), which is a new form of interdisciplinarity between physics and mathematics, referred to as a phenomenotechnology by G. Bachelard.

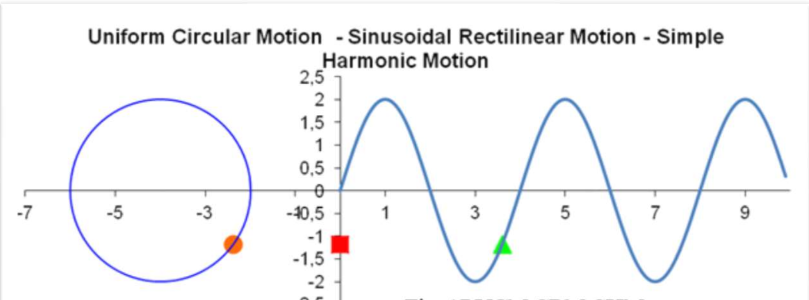


Fig. 17 UCM-SRM-SHM

Using derivatives of trigonometric functions, the type of force that gives rise to a simple harmonic motion is characterized, proportional to the extension and restoration. The link between the simple pendulum (SRM) and simple harmonic motion is not rigorously proven. They are based on an inductive inference, either by observing the curve described by a funnel filled with sand in pendular motion on a piece of paper moving in URM (Fig. 18), or by analogy between the projection of a uniform circular motion on one axis (Fig. 19) and the projection of the movement of the pendulum over the ground.

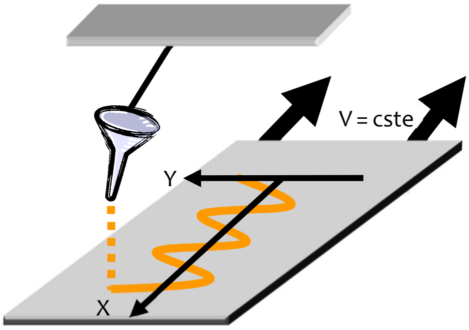


Fig. 18

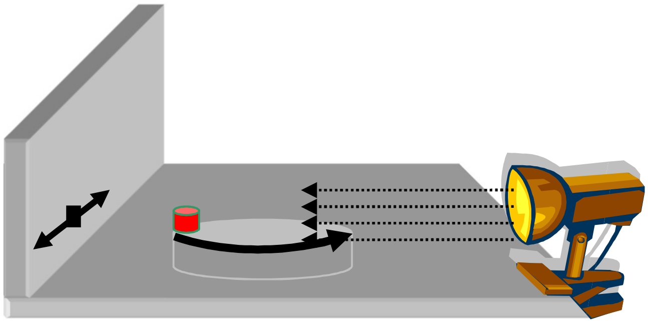


Fig. 19

So why, when the students know how to resolve forces and know Newton's law of mechanics, do they not make use of the associated ODE? This problem also arises during the first year at the university.

For Benson, the situation is even more complex. In fact, the author goes around in circles. He starts from general harmonic motion y(t) = A sin (ω t + φ) and finds the first and second derivatives to compare position and acceleration, thereby obtaining the ODE. He then asserts that the extension is **one** solution for this ODE. He does not prove the uniqueness of the solution, but above all he sets out from observation of block spring movement to state that its elongation is sinusoidal.

Yet it would be sufficient to set out Newton's law, use linearization and find a function whose second derivative is opposite to itself. The uniqueness of the solution would then have to be proven. This proof can be provided on the basis of mathematical theorems encountered in the final years of secondary school. It is a shame that ODEs are not linked to the law of mechanics in some universities.

We also note that at the Université catholique de Louvain, engineers and physicists no longer have a compulsory course in ODEs. A number of modelling techniques that make significant advances possible in physics and in understanding of physical phenomena are, however, based on ODEs and partial differential equations (PDEs).

At the university level, it would also be useful to work alongside students on questioning the error that is made when the simple pendulum is made linear. In many works on physics, there is reference to a margin of error without providing any explanation. Is this the error made in comparing sin(x) and x or the error made in comparing the solution of the linear equation with that of the non-linear one? To answer this question, we invite the reader to consider, for example, the application of Grönwall's lemma in N. Rouche and J. Mawhin (1973) to find a maximum bound for this error. We also find this lemma in Chapter 5 of Demailly's book, but it is not used to obtain a bound for the error that is made. The error is calculated using asymptotic methods in Chapter 11. It would therefore be interesting to analyze the dual approaches used in these two methods.

Conclusion

A reflection on the use of dx/dt as a ratio and the value of such an incorrect course is not adequately taken into account in teaching, nor is the problem of dx, which is infinitely small. Nevertheless, this is an epistemological obstacle, as described by Kac and Randolph (1942) and by Poincaré (1904) and Hadamard (1923). In fact, the notations $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = x'$ and $dx = x'\,dt$ do not refer to the same theoretical framework, since one refers to the derivative and the other to the differential.

In secondary school, the teaching of analysis is based on classical analysis. It is the same at the beginning of the university course. The use of the notion of limit refers to the analytical framework that does not allow to interpret the differential quotient as a real ratio, but rather as a function deriving from the first one. From our epistemological study of the birth of infinitesimal calculus, we have shown the emergence of two currents. Leibniz, founder of the differential quotient rating, expressed doubts about the advisability of such a rating by offering a functional rating. Then, with the advent of the concept of limit, the algebraic framework has gradually disappeared.

During the acute crisis of the foundations at the beginning of the twentieth century appeared a questioning of classical analysis with, for example, the appearance of constructive mathematics. It would have been too long to mention in this article the ups and downs of this crisis. Moreover, we think that, as N. Rouche (1995) wrote, the interest for the global structure and for the questions of foundation, very strong in the beginning as well as in the middle of the 20th century, seems to weaken.

We believe that, in light of the study of the difficulties encountered by pupils and students in this epistemological obstacle, referring to unknown theories of learners is very risky and harmful. Justifying the differential quotient, as a real report after having obtained it by the limit defined in classical analysis, relying on unknown theories of pupils and students at this stage of their curricular course, amounts to asking them to be credulous and lose their critical thinking. This differential quotient defined by the limit resulting from the classical analysis is not a true ratio.

On the other hand, using a ratio, in a defined algebraic framework, makes it possible to solve problems using the concept of relative velocity. At the same time, the use of a function, in an analytical framework, makes it possible to solve problems using the concept of instantaneous velocity. The result is a dual approach to problems, which allows each student to adopt a method for solving them that suits his or her own way of thinking.

This dual approach also makes it possible to adapt to the students' learning process, taking their knowledge into account. It is not possible in secondary school to use deductive inference to find the formula for centripetal acceleration, but inductive inference is used, based on observations of specific phenomena. The concept of limit must be avoided before it has been addressed, constructed and defined during several courses. To explain it, the second example of kinematics of applied duality is a good source of inspiration. It refers to the concept of tangent, minimum distance, instantaneous speed, double root, relative speed, [...] and finally limit.

That is why this article has addressed the comparative study of the approaches to concepts, problems and applications used by mathematicians and physicists. It is important for future teachers to be aware of different approaches during their training so that they can use them in teaching practice by introducing changes of framework, allowing students to understand the problems better. At Paris-Diderot University, a dialogue between the various lecturers in the two disciplines has been taking place to prevent ambiguities arising during classes and causing additional difficulties for students. Joint sessions led by a mathematics lecturer and a physics lecturer and demonstrating the complementarity of the different approaches, were arranged for students on a variety of topics. Unfortunately, this attempt proved unfruitful in the case of the session on the differential, since an agreement could not be reached. In conclusion, we are still a very long way from finding any synergy that is didactically beneficial. The interpretation of the derivative in terms of ratios is a true epistemological obstacle that hampers progress, and unfortunately does so in both disciplines.

Glossary

A **framework** consists of a set of objects from a branch of mathematics, relationships between these objects, the possibly diverse ways they are formulated and the mental images associated with these objects and relationships. (Douady 1984 and 1992)

The **algebraic framework** refers to the following objects: the dimensional or non-dimensional ratios of numbers represented by letters (delta x, dx, a, v, etc.) and the associated algebraic operations.

The **analytical framework** refers to the following objects: numerical functions and operations on these.

A **change of framework** is a means of obtaining various formulations of a problem, which, without necessarily being equivalent, allow a new approach to the difficulties encountered and the application of tools and techniques that were not imperative in the first formulation.

A **conceptual window** is the set of objects, tools and relationships that are used to analyze the terms of the problem or situation, or to develop a problem-solving strategy, whatever the frameworks under which they come. (Douady 1992)

An **inference** is a mental process that consists of selecting components of a complex entity, keeping some and ignoring others. This is true for both inductive reasoning and deductive reasoning. (Barth 1987)

Inductive inference infers a rule from limited information, based on the observation of particular facts and examples. This method is used to infer from the particular to the general (theoretical). These are hypotheses or conjectures, which must then be validated.

Deductive inference is a precise conclusion based on a given truth. This method is used to infer from the general or theoretical to the particular. The inference is necessarily true.

Interdisciplinarity between physics and mathematics is a tool for curriculum development, pedagogy and didactics that seeks, within one of the two disciplines, mathematics or physics, to build and integrate knowledge, adopt concepts and model reality on the basis of discipline-specific knowledge from the other discipline. It involves collaboration between mathematicians and physicists.

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