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Article

The Concept of Function at the Beginning of the 20th Century: A Historiographical Approach

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Abstract:
The evolution of the concept of function at the beginning of the 20th century in France after the definitions by Dirichlet and Riemann and the introduction of several pathological functions is studied. Some young mathematicians of those years (Baire, known for his classification of discontinuous functions, Borel and Lebesgue famous for their new theories on measure and integration) made several attempts to propose a large class of functions as "accessible" objects. Their discussions, their purposes and polemics are reported often by their own words supported by a large bibliography. The contribution of some Italian mathematicians, as Vitali, is also underlined. Some of such discussions are linked to the growth of measure and function theories, others will find mathematical answers in the modern theory of computability for real functions.

Keywords:
Borel and Lebesgue measurable functions; Baire classes; Borel–Lebesgue controversy; Axiom Choice; Lebesgue not Borel measurable functions

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> E ... si dà luogo... alla domanda "se, conservando tutta la generalità contenuta nella definizione, una funzione y di x data in un certo intervallo possa sempre o no esprimersi analiticamente per tutti i valori della variabile nell'intervallo stesso per una o più serie finite o infinite di operazioni di calcolo da farsi sulla variabile", e a questa domanda, nello stato attuale della scienza, non può ancora rispondersi in modo pienamente soddisfacente, poiché, quantunque si sappia ora che per estesissime classi di funzioni e anche per funzioni che presentano grandissime singolarità può darsi un'espressione analitica, resta però ancora il dubbio che, non facendo nessuna limitazione, possano anche esistere funzioni per le quali ogni espressione analitica, almeno con gli attuali segni dell'analisi, è del tutto impossibile. (Dini 1878, 37)[2]

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[2] And ... the question arises "whether any function *y* of *x* defined in an interval could always be analytically represented for all the values of the variable in that interval by means of one or more finite or infinite series of operations to be done on the variable, or not"; in the present state of science the

Introduction and Bibliography

In the cultural ferment of the first decades of the 20th century the following mathematical problems went through a rigorous examination: What is a "genuine" real function? How can it be defined, described, identified? What functions are effectively accessible to the mathematicians? As an intuitive geometrical concept of the analysis was abandoned and new objects without physical reference burst into the mathematical practice, whether they were sets or functions, it was necessary to introduce suitable methods to access to these new entities. Indeed, during the whole of the 19th century almost all mathematicians, grounded on an intuitive geometric outlook on mathematics, thought that analytic functions, that is functions that are sum of a power series, were more than enough for all purposes. Precisely, a real function f in an interval $]a,b[$ is called *analytic* if it is such that for every $x_o \in]a,b[$ there exists a real number $\delta>0$ and a sequence of real numbers $a_o, a_1, a_2, \ldots$ such that

$$f(x)=\Sigma a_n(x-x_o)^n \text{ for every } x \in]x_o-\delta, x_o+\delta[.$$

An analytic function is very regular, i.e. indefinitely derivable. All the elementary functions, polynomials, sinx, cosx, e^x, sinx, cosx, logx … and the functions obtained from them by means of the algebraic and the composition operations are analytic, but there are also analytic functions without representation in terms of elementary functions. However, in relation with the solution of the heat differential equation in 1822, in his paper *Théorie analitique de la chaleur*, J. Fourier (1768-1830) claimed that "arbitrary", more general functions could be represented by a trigonometric series. Just by studying Fourier series, in 1837 Dirichlet (1825-1859) considered the totally discontinuous function whose value is 1 for rational and 0 for irrational arguments (the Dirichlet function) and furnished the first "abstract" definition of a real function of a real variable. This definition was well accepted: it closed at the time a long dispute about what had to be understood by the word function, in which many mathematicians had been involved. The dispute had begun after the publication of D'Alembert's paper about the vibrating chord in 1747: indeed, d'Alembert thought that the solution, even if not necessarily analytic, had to be sufficiently regular in every time, because it was a solution of a differential equation. On the contrary, Euler, in 1748, in *De chordarum exercitatio,* exposed his thought that the starting position could be arbitrary, the presence of one or more angular points signifying that in those points the curve had been pinched. As a consequence, Euler, in 1755, in his *Institutiones calculis differentialis*, had given a general definition, trying to enclose all possible ways in which a quantity may be determined by other quantities: precisely he defined a function as a quantity which depends on other quantitities in such a manner that, when those vary, it also varies. A progress with respect to the definition appeared on *Introductio in analysin infinitorum* in 1748 where a function consisted in an analytic expression of a variable quantity, in other words, a sum of a power series.

After Dirichlet's definition of abstract function, other facts came to light: Riemann (1826-1866) furnished an example of an integrable function with an infinite number of discontinuities between any two limits. He defined (x) as the excess of x over the nearest integer, that is $(x)=0$ if $x=|x|+½$; $(x)=x-|x|$ if $x<|x|+1/2$; $(x)=x-|x|-1$ if $x>|x|+1/2$, and therefore $-1/2<(x)<1/2$ for every x. Then he considered the following function:

question cannot be answered in a completely satisfactory way since, notwithstanding very large classes of functions and even many very singular functions are known to be analytically representable, it is not clear yet, at least by the current means of analysis, if there exists any function without analytical representation (Dini 1878).

$$f(x)= \frac{(x)}{1} + \frac{(2x)}{4} + \frac{(3x)}{9} + \dots = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$$

The series converges for every x. All the points $x=p/2n$ with p and n relative prime are discontinuity points with $f(x+)$ and $f(x-)$ finite and such that the jump in these points decreases to 0 when n diverges. The function $f(x)$ is continuous in all the other points. Therefore, by Riemann integrability condition the function is integrable. If we consider its integral function $F(x)=\int f(x)dx$ we obtain a continuous function that is not derivable in all the discontinuity points of $f(x)$ and therefore fails to have derivative at infinite points in every interval. Riemann discussed the function $F(x)$ in his inaugural dissertation in 1854, but it was published only in 1867 (Riemann 1898). Notwithstanding $F(x)$ is pathological in the sense it is without derivative in an infinite number of points and therefore it is not possible to visualize it, it was well accepted, since such points constitute only a subset of the rational number set.

The situation was quite different for the function presented in 1861 in his lessons at the University of Berlin by Weierstrass (1815-1897) that caused a complete revision of the concepts on the basis of mathematical analysis. Weierstrass gave the example of a continuous function in the real axis, but without derivative at any point. The surprising function has the following formulation:

$$f(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x),$$

where a is an odd integer and $b \in]0,1[$ is such that $ab>1+3\pi/2$[3].

G. Darboux (1842-1917), in his famous memoir (Darboux 1875), gave an example of another pathological continuous function that is neither increasing nor decreasing in every interval: also, this function cannot be expressed in terms of a power series and it is impossible to visualize it.

As a consequence, discussions arose whether general functions represented by trigonometric series or only analytic functions could be useful for mathematicians: what significance could have pathological functions or functions without an analytical representation? Was the horror felt by Hermite (1822-1901) for continuous functions without derivative justified?

The situation is well summarized in the last years of the 19th century by G. Frege (1848-1925) in (Frege 1960). He noticed that the reference of the word "function" had been extended by the progress of science. Indeed, the field of mathematical operations that serve for constructing functions had been widened by adding to the traditional operations of addition, multiplication, exponentiation and their converses, the various means of transition to the limit, thus adopting something essentially new. Moreover "People […] have actually been obliged to resort to ordinary language [...], e. g., when they are speaking of a function whose value is *1* for rational and *0* for irrational arguments" (Frege 1960, 28).

What is completely new, Frege goes further still and admits not necessarily mathematical objects as arguments and values of functions. He calls "the capital of x" as the expression of a function and says that if we take the German Empire as the argument, we get Berlin as the value of the function (Frege 1960, 31).[4]

To conclude, in the early years of the 20th century, after the work of Weierstrass, Darboux and Cantor the situation was less confused, but undoubtedly more varied. While, in general, many mathematicians were not interested in the subject and some of them kept

[3] A reader can find a deep analysis of this period in (Manheim 1964).

[4] But which are the admitted objects? Frege only says that an object is anything that is not a function, so that an expression for it does not contain any empty place. A statement contains no empty place and therefore we must regard what it stands for as an object. But what a statement stands for is a truth value, true or false. Thus, Frege allows a function that leads from a proposition to its truth value.

even a hostile attitude,[5] only a few of them tried to give an answer the previous questions, with different motivations, some of them revolutionizing at the same time measure and integration theory and launching the modern theory of functions of a real variable.

This paper will be focused on the thorough analysis about the ontological character of the mathematical concept of function, and the manner to do mathematics in which many mathematicians, particularly French, as E. Borel (1871-1956), R. Baire (1874-1932), H. Lebesgue (1875-1941) and M. Fréchet (1878-1973), but also Italian, as G. Vitali (1875-1932), G. Fubini (1879-1943), Beppo Levi (1875-1961) and L. Tonelli (1885-1946), were involved.

The reader can find some contact points with the similar analysis developed for example by Monna (Monna 1972). H. Gispert in (Gispert 1995) studies deeply the work of the French mathematicians in connexion with the development of the set theory by G. Cantor but she is not interested in the disputes and controversies risen after the publication in 1904 of Zermelo's paper about the Choice Axiom. This period is thoroughly analyzed in this paper. Other suggestions and information about the analysts which developed their work using the empiristic theory of sets after the apparition of Zermelo's paper can be found in (Cavaillès 1937) and in (Cassinet and Guillemot 1983).

The contribution of some Italian scholars to the growth of a new way to understand mathematics will be also underlined: indeed, in particular the thought of U. Dini (1845-1918), one of the most important Italian mathematicians of XIX century, was characterized by a deep interest in some theoretical questions about real analysis and measure theory; on this subject we have to notice that, only three years after the fundamental memory by G. Darboux (Darboux 1875), U. Dini dedicates a whole chapter of (Dini 1878) to the Hankel *singularities condensation principle*, giving a rigorous proof of it and many interesting examples of singular functions. By this principle, starting with a function that is singular with respect to continuity or derivative in one point only, it is possible to create analytical expressions of infinitely many functions which present the same kind of singularity in infinite points in every part of the interval where they are considered. It is also possible to determine the analytic forms of the Dirichlet function, that will be the object of a paper of a disciple of U. Dini, Alberto Tonelli (Tonelli 1885) to be not mistaken for Leonida. Also the chapter, "Funzioni che non hanno mai la derivata determinata e finite", of (Dini 1878,147-166), deals with deeply pathological functions.

As another example, the interest in the study of the analytic representation of discontinuous functions is proven also, some years after, by the early paper by Severini (Severini 1897), where the Weierstrass approximation theorem is extended to a class of functions which are Riemann integrable and therefore can present a particular kind of singularities.

But there are many other Italian mathematicians that gave a fundamental contribution towards real analysis in the second half of 19th century: we limit ourselves to mention G. Ascoli (1843-1896), C. Arzelà (1847-1912), V. Volterra (1860-1940), who was one of the founders of Functional Analysis, and G. Peano (1858-1932): in 1887 this former author, introduced for the first time Peano measure in his treatise (Peano 1887). Only in 1893, in the second edition of the first vol. of his *Course d'Analyse*, (Jordan 1893), C. Jordan will give the same definition for subsets of a Euclidean space.

Informations about the Italian authors of the fist decades of the 20th century and their contribution to the development of measure and function theory can be find in the papers (Pepe 1984), the Preface of G. Pepe to Vitali (Vitali 1984) and (Vaz Ferreira 1991).

[5] See for example (Pepe 1984), the Preface by G. Pepe to Vitali (Vitali 1984, 8-11) and (Vaz Ferreira 1991, 377-78).

The First Papers on Real Analysis at the Beginning of 1900

In the years around 1900, Baire, Borel, Lebesgue were the mainly involved mathematicians in a deep analysis and some hard polemics about the concept of function.

Baire was the first to be interested in real functions in an exclusive manner. Lebesgue wrote about him, in the *Comptes rendus des séance de l'Académie des Sciences* (Lebesgue 1958, 67), immediately after his untimely death in 1932 that while before Baire the scholars were interested in real variables only by chance and for the study of complex analysis, their exclusive occupation since the beginning of the 20th century, Baire was the first to devote all his scientific activity to the theory of real variables.

Baire observed (Baire, 1899 and 1905) that in many applications we have often to consider discontinuous phenomena. Every kind of singularities and discontinuities can enter in many questions beyond the mathematician's control, hence the scholar has to study in an abstract manner the relations between the two notions of continuity and discontinuity; but often the mathematicians pass over a general treatment of the problems, considering only particular cases. So, with respect to the functions, they consider only analytical functions or more generally derivable functions, studying their properties deeply. Baire thought on the contrary that it is worthwhile doing a more general approach to mathematics, determining what functions are accessible to our study. He wrote that in a course of classical Analysis the fundamental notions are presented immediately in a very general manner; straight afterward the field of study is restricted by imposing some restraints. Thanks to them it is possible to construct the theories which constitute Mathematics. It is therefore justifiable to try if it is possible to deduce from the first general definitions all the general conclusions. In this way one can propose to develop beyond the current Analysis, another branch of Analysis, different from the first with regard to the quantity of obtained results, but on the other hand offering more complete enunciates (Baire 1905, III).

In this context, it is important to answer the following question: under what conditions is a discontinuous function the limit of a sequence of continuous functions? The treatment of this problem occupies largely his work. The answer is known as Baire Theorem:

> *A real discontinuous function of a real variable is the limit of a sequence of continuous functions if and only if it is punctually discontinuous, that is its continuity points are dense in every perfect subset of the real line.*

In his Thesis, Baire (Baire 1899) develops his program exposing a subdivision into classes of all functions (Baire's functions) which should be of interest to mathematicians: in the class 0 there are the continuous functions, in the class 1 the functions that are limits of continuous functions but are not continuous, in the class 2 the functions that are limits of functions of class 1 but are not in class 1 and so on, up to classes of transfinite order.[6]

One year before, the book (Borel 1898) had been published: a new fundamental approach to measure theory came into sight. Sets that have the power of continuum as the Cantor set[7] or are dense on an interval as the rational number set have null measure! A constructive attitude with respect to mathematical objects became a prevalent habit and the question of the existence and the definition of sets and functions an essential question. As

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[6] Good sources of information about Baire are (Gispert 1995) and (Dugac 1975-76).

[7] The Cantor set on the closed interval [0,1] is constructed in the following way: we first estract from [0,1] the middle third, i.e., the open interval]1/3, 2/3[of lenght 1/3. We then proceed similarly with each of the two remaining intervals, i.e., we estract from each of them the middle third, namely, the open intervals]1/9, 2/9[from [0,1/3] and]7/9,8/9[from [2/3,1]. An analogous procedure is carried out with each of the four remaining closed intervals and the process is continued indefinitely. The closed set which results is called Cantor set. It has zero measure and the power of the continuum.

an example, what is the meaning of a genuine function? Such for Borel is an *effectively defined* function that is a function such that it is possible to calculate, by means of a limited number of operations and with a given approximation, its value corresponding to a given value of the variable (Borel 1898, Note II, 117).

Another question, linked to the previous one is the following: there are no problems with integer numbers or geometrical continuum, of which we have an immediate intuition, but can we have a concept of sets whose power is greater than the continuoum? Can we have a general conception of discontinuous functions?

As a consequence of his ideas about mathematics, Borel's answer is no, as he thinks that in the practice the use of these concepts is difficult and sometimes impossible, and the mathematicians have to use *effectively enumerable* sets and well-defined sets and functions.

Borel believes that it is difficult to introduce the consideration of all the classes of functions whose set has a power greater than the continuum because we are not able to calculate a function if it is not defined by means of at most infinitely enumerable many elements. This happens for continuous[8] functions or functions having only an enumerable set of discontinuities (Borel 1898, Note I, 109). On the contrary a discontinuous function is defined by *infinitely not enumerable many conditions*; to all practical purposes it is impossible to define it (Borel 1898, Note III, 126).

Summarizing it is necessary to distinguish, both in the sets of points and in the discontinuous functions two big classes: the sets and the functions which cannot be defined by an enumerable set of conditions and those which can defined in such a way: only the latter can be considered useful (Borel 1898, Note I, 110).

Few years after, in his *Leçons sur l'intégration*, (Lebesgue 1904a), whose main purpose is a revolutionary integration theory, Lebesgue introduces new arguments for the discussion. He thinks, as Baire, that discontinuous functions play an important role in mathematics, but he does not limit himself to the Baire functions: his theory of measurable sets and functions is modelled according this idea and shares not much with Borel measurable sets and functions (he calls B measurable). Lebesgue feels however to be in debt to Borel for his measure theory and in the first time he tries to follow as much as possible Borel's point of view. So he observes (Lebesgue 1904a, 109, footnote 1) that the set of the B measurable sets has the continuum power, therefore there exist other sets besides the B measurable sets; this does not mean however that it is possible to exhibit a not B measurable set, that is to say a finite number of words characterizing one and only one not B measurable set. He says that we will always meet only B measurable sets. In particular, all Baire functions are B measurable.

So the field of the functions that could interest the mathematicians had become very wide and new classes of functions attracted their attention: Baire's functions, effectively defined Borel functions, Borel and Lebesgue measurable functions.

On the basis of the conception of these classes there were, as we will see, some general ideas that even if not strictly related to the intuitionist philosophy, shared its constructive approach and are generally known under the name of French empirism. The methods that would be used were characterized by two fundamental criterions: they had not to imply contradictions and they had to be accessible to mathematicians and even available for applications.

[8] Indeed, Borel considers, as Baire, real functions of real variable and observes that, in order to define a continuous function, it is enough to consider the values it assumes in the rational numbers.

The Analytically Expressible Functions by Lebesgue

In (Lebesgue 1905) the author introduces the analytically expressible functions, that is the functions that can be built from polynomials by the ordinary arithmetic operations and by passing to the limit a finite or enumerable number of times.

Accordingly, all the elementary functions, being analytic, are also analytically expressible, but the converse does not hold. Indeed, since the limit of a sequence of analytic functions, in general, is not analytic, it is obvious that the new definition is very wider than the one given by Euler in 1748 in his *Introductio in analysin infinitorum* where all the defined functions were analytic. The totally discontinuous Dirichlet function is not analytic but it is analytically expressible since:

$$\chi(x)=lim_{m\to\infty}[lim_{n\to\infty}(cos\, m!\, \pi x)^{2n}].$$

Lebesgue treatment is very close to the study of function classes done by Baire some year before; we are not astonished at the fact that one of the basic results of the paper is the following theorem:

A function is analytically expressible if and only if it belongs to a suitable Baire class.

The proof is immediately drafted: every continuous function is analytically expressible by the Weierstrass theorem (it is the uniform limit of a sequence of polynomials). The other Baire functions can be obtained from continuous functions passing to the limit once or more times and therefore they are analytically expressible too. The converse is trivial because all the polynomials are continuous.

Then Lebesgue classifies the Borel functions, proving that:

A function is analytically expressible if and only if it is Borel measurable.

An obvious consequence is that:

A function is Borel measurable if and only if it belongs to a suitable Baire class.

We can read the proof of this equivalence also in a paper of the same year 1905 by Vitali, *Un contributo all'analisi delle funzioni,* (Vitali 1984, 189), where also the concept of truncation of a function appears, but not with regard to integration as will be considered by de la Vallée Poussin.

We ought to observe that in 1905 G. Vitali had yet wrote a brief paper, *Una proprietà delle funzioni misurabili* [9](Vitali 1984,183), where he proved that

Every real measurable function is the sum of a function of class not greater than 2 and of an a.e. zero function.

There was some interest in the nature of Baire functions in those years: for example, in a postcard of June 1st, 1905 Guido Fubini writes to Vitali, (Vitali 1984, 454) and with regard to the previous paper, claims that all Baire functions are measurable, since limits of

[9] A first version of this paper was written since 1903, as follows from the letter of Pincherle to Vitali dated November 22th 1903 (Vitali 1984, 435). In this letter, Pincherle gives advise about the draft of the paper that in his opinion contains an interesting result.

measurable functions. So they are the sum of a second class function and a function with null integral.[10]

Fubini comes back to this subject by another postcard on June,12st.

Also, Lebesgue in his thesis (Lebesgue 1902) had already established that all Baire functions are just summable (but, in that context, the meaning of summable is the same of measurable); moreover, in (Lebesgue 1904b) he had deeply studied several properties of first class functions.

Here the proof of the previous result we can find in (de la Vallée Poussin 1916) is reported because, notwithstanding it dates back to some year after, it is written in a very clear style.

First, de la Vallée Poussin proves, as Lebesgue in the *Leçons*, that Baire functions are B measurable: indeed, the continuous functions are B measurable and all the other Baire functions are obtained from them by means of limiting passages: but the limit of a sequence of B measurable functions is a B measurable function.

Then he proves the converse implication by steps. First of all the characteristic function of a closed set *D* is of class 1: indeed, the function associating with every point *P* of the space its distance from *D*, *r*, is continuous and is zero if and only if $P \in D$, therefore the characteristic function of *D* is given by the limit for *n* tending to infinity of the sequence of continuous functions $\frac{1}{1+nr}$ and therefore it is a Baire function of class *1*. Consider now whatever B set *T*: since *T* can be obtained by finite or enumerable unions or complements of closed rectangular domains, its characteristic function is a Baire function. We now can conclude that every B measurable function *f* is a Baire function since it is the limit of a sequence of linear combinations of characteristic functions of B measurable sets. Indeed if, for example, *f* is bounded, its set of values is enclosed in an interval, this interval can be divided by the points $a_1<a_2< \ldots <a_{n+1}$ such that $a_{i+1}-a_i<\varepsilon$ for $i=1,2, \ldots,n$. Let E_i be the Borel set of the points *x* such that $a_i \leq f(x)<a_{i+1}$ and let f_i be the characteristic function of E_i. Then *f* is the limit for ε tending to zero of the Baire functions $a_1f_1+a_2f_2+ \ldots +a_nf_n$ and therefore *f* is a Baire function.

Since every Lebesgue measurable function coincides a.e. with a B measurable function, every Lebesgue measurable function a.e. coincides with a Baire function. In this way the universe of all the functions that can be of some interest for the mathematicians could be described by Baire functions or analytically expressible functions and a.e. null functions.

Let us return to (Lebesgue 1905); Lebesgue proves again many technical results, some of them already proved by Baire, for example:

An analytically expressible function is punctually discontinuous on every perfect set, when the sets of first category[11] *with respect to such a perfect set are ignored.*[12]

Lebesgue hints at the possibility that this necessary condition is also sufficient but does not face the question (Lebesgue 1905, 188).[13] Instead, he proves a theorem that furnishes a new way to define the functions of class *n* by continuous functions:

> *The Baire real functions of class n coincide with the functions of n+1 variables that are continuous with respect to every variable when the same value is given to all variables.*

[10] Such a function is a.e. zero by another paper by Vitali of the same year, *Sulle funzioni ad integrale nullo* (Vitali 1984, 193).

[11] Given a metric space *X* a subset *E* of *X* is called a first category set if it is the union of a sequence of sets such that the interior of the closure of every one is the empty set. The other sets are all of second category following Baire's nomenclature.

[12] As an example, the Dirichlet function is continuous on the set we obtain from the interval [0,1] ignoring the rational number set (that is a first category set).

[13] In (Lusin 1930, *154-155*) the answer is negative.

Finally, in the last chapter of his long memoir, Lebesgue proves that in every Baire class there exists some element and that there exist functions which are not analytically expressible.

In this context Lebesgue claims that he will try to never talk about a function without defining it effectively; so, he is very near to Borel's position. For example, Lebesgue, as Borel, considers a function as perfectly defined if it is possible to denote it by a finite number of words. But he advises: no wonder if later he will consider as completely defined functions he is unable to calculate for any value of the variable.

The Dirichlet function is of this kind, indeed it is, as we have already seen, analytically expressible and therefore B measurable, *bien-définie* in the meaning of Borel, but it is totally discontinuous and therefore if *x=C*, where *C* is the Euler's constant,[14] since we do not know whether it is a rational number or not, it is impossible to calculate the value of the function in this point, even if it is possible to calculate so many decimal ciphers of *C* as we want.

This example is important in the evolution of Borel's thought: some years after he will call *asymptotically* defined[15] a function like Dirichlet's function. In 1912 he says that a bounded function *f* defined in a domain *D* is *asymptotically equivalent to polynomials* if there exists a sequence of polynomials converging in measure to *f*. Then he proves that the limit of a bounded sequence of functions that are asymptotically equivalent to polynomials is asymptotically equivalent to polynomials and therefore every analytically expressible function is asymptotically equivalent to polynomials (Borel 1912, 192-193). It is worth noticing that the concept of converge in measure was known by Borel and Lebesgue since 1903 (Lebesgue 1903, 1229); see also further development in (Lebesgue 1906,9), and (Borel 1905, 35). In (Lebesgue 1903) the author for the first time mentions the topological property of a measurable function, that is if *f(x)* is a measurable function defined in the interval *[a,b]*, then for every $\varepsilon>0$ there exists a set *H* such that *f(x)* is continuous in the set *[a,b]-H*. This property will be clearly stated and proved by Vitali in (Vitali 1905, 183). In the same year 1905 Vitali will make clear the content of a note by Lebesgue to his *Leçons* (Lebesgue 1904,129), giving the definition of *absolutely continuous function* and proving that a function is an integral function if and only if it is absolutely continuous.

Let us come back to (Lebesgue 1905); as we have seen Lebesgue shifts the question of the definition of mathematical objects from the possibility of their effective construction to a "language" problem: a mathematical object is defined if it is possible to describe it with a finite number of words that apply to the object in an unequivocal way, that is when a characteristic property of the object can be *named;* for example the function, equal to one if the Euler's constant C is rational, equal to zero otherwise, is named, in other words is

[14] The Euler's constant C is defined as the limit of the converging sequence: $\sum_{k=1}^{n}\frac{1}{k}-\log n$.

[15] In (Borel 1912) the author is definitely against the analytical definitions: indeed, he considers a definition like "the number *a* is equal to zero if the Euler's constant is an algebraic number and one otherwise" at the borderline of mathematics. In it, the defined number depends on some unknown event: the definition is regarded as a mathematical one only for the mathematical nature of the unknown event and therefore it is possible to give the number *a* an analytic definition. But in order to obtain an explicit formulation for it, many passages to the limit are necessary so it is really impossible to calculate it. Then the analytic definition does not have any mathematical meaning, but it is merely the translation in a more complex language of the primitive definition in such a way that eventually the value of the number *a* depends on the solution of a problem that we are not able to treat by any regular method. The fact that the problem is mathematical in nature seems to be an accessory circumstance.

defined, even if we do not know a method in order to state with a calculation what is the effective value it assumes.[16]

Such a kind of definition of the mathematical objects is the outcome of a continuous examination about their nature: one year before, in the *Leçons*, Lebesgue espouses a point of view like Hilbert, when he begins Chap. VII making a list of the six properties the integral has to verify; he writes that in this way he is defining it, the definition being descriptive and not constructive. A descriptive definition requires that the enunciated conditions have to be compatible. In a footnote, he adds that the aim of the *Leçons* is just searching a constructive definition of the primitive functions equivalent to the descriptive one, that is a model of the given system of axioms in order to prove also its consistency. Here the constructive attitude of Lebesgue obviously does not coincide with the stronger point of view of Borel.

But, as we have seen, in 1905, Lebesgue drops Hilbert's approach taking up the new idea, a vague criterion of almost extra-mathematical character in the opinion of Borel.

Lebesgue does not agree to carry out infinitely many choices without determining the law they are made out. It is worthwhile observing that the axiom of choice is the object in 1907 of an interesting paper (Lebesgue 1907), where Lebesgue studies for the first time *functional operations*.

He considers in place of the subsets of a given set the characteristic functions of their complements, in such a way the family of the subsets of a given set can be seen as a family of functions $f(x,\lambda)$ depending on a parameter λ and the Zermelo function can be seen as a correspondence that associates with every λ a number $y(\lambda)$ such that $f(y(\lambda),\lambda)=0$.

As Lebesgue observes, in the early years of the 20th century, several functional operations were known, as for example the l.u.b. or the g.l.b. of a set or of a function, the total variation, etc..., or the correspondences studied by Volterra, Hadamard, Pincherle[17], but they had not been sufficiently studied yet, except the definite integral.

Lebesgue tries to give a definition of the previous operation, as an analytical procedure associating an analytically expressible number $y(\lambda)$ with every analytically defined function $f(x,\lambda)$. He proves that even if we limit ourselves to the enumerable sets it is impossible to define for them a Zermelo correspondence by means of analytic procedures. In other words, Lebesgue proves that no analytically representable nor only measurable function exists defined in an enumerable set X such that with every subset $\{x_1, x_2, x_3, \ldots\}$ of X associates a number from $\{x_1, x_2, x_3, \ldots\}$.

By this proposition Lebesgue agrees with Borel's point of view about enumerable sets: Borel writes in (Borel 1905,164) that, in general, we have to distinguish between enumerable sets and effectively enumerable sets since a proper part of an effectively enumerable set can be not effectively enumerable and all the supposed paradoxes of the set theory are caused by the wrong proposition: every enumerable set is an effectively enumerable set.

Coming back to (Lebesgue 1905), Lebesgue proves that it is possible, for every finite or transfinite number α, *nommer* a function of class α. Borel faces the same question the same year, in the Note III in the appendix of (Borel 1905); since the set E of the functions whose class exceeds a given number has the power of the continuum[18] while, by Cantor's theorem

[16] It is also possible to name a set such that we cannot decide whether a given object belongs to it or not. For example, let E be the singleton $\{1\}$ if the Euler's constant is rational and let E be the empty set otherwise: then it is not possible to decide whether *1* belongs to E or not.

[17] The book (Pincherle 1901) was a pioneering work on Functional Analysis. In it a fundamentally new point of view is exposed: the analytic functions are conceived as points in an infinite dimensional linear space and the theory of functions is the study of linear functionals on this space. In the following years, Pincherle wrote some other papers about this topic but only some years after they seemed outdated owing to the rapid and remarkable growth of the newly born theory of Functional Analysis.

[18] In (Lebesgue 1905) the author observes explicitly that the family of the α class functions, since it is defined by an enumerable set of conditions, has the power of the continuum in accordance with Note

the set *F* of all functions has power greater than the continuum, the set *F* contains infinitely many functions that do not belong to *E*. Borel claims that this reasoning, based on the power, has a serious fault: we know that there are functions that do not belong to *E*, but we have not the manner *to define one of them*, that is to designate one of them in such a way to be able to distinguish it from the others.

Then Borel furnishes a procedure to prove that there exist functions that are not in the classes *0,1* and *2* trying to exhibit one of them; he claims that such a procedure can be generalized to classes of greater order, but notices that the arising definitions will imply some impracticable operations.

Here is the proof he gives: since the set of the functions belonging to the classes *0, 1* and *2* has the power of the continuum, there is a one-to-one correspondence between it and the set of numbers of the real interval $(0,1)$ which is possible to exhibit in a *bien déterminée* way. Then let f_x be the function corresponding to the real number $x \in (0,1)$. We can consider the following function $F(x)=0$ if $f_x(x) \neq 0$; $F(x)=1$ if $f_x(x)=0$. Of course such a function does not belong to the previous classes, but, as Borel notices, it could be also beyond any classification. He remembers that he had communicated this proof to Lebesgue who had improved it and obtained the definition of a function certainly of class *3* to appear on the Jordan Journal.

Borel obviously referred to (Lebesgue 1905) where Lebesgue proves that for every ordinal α it is possible to name a function belonging exactly to the α class; moreover in it Lebesgue proves that there are functions that are not analytically expressible and therefore are not B measurable, but one can name.[19]

Thus, besides B measurable functions whose values we are not able to calculate, but we can name, like Dirichlet function, there are also Lebesgue but not B measurable functions and sets that can be named:[20] Lebesgue considers a set of this kind (Lebesgue but not B measurable), contained in a perfect one whose measure is zero[21] and observes in a footnote that every bounded function that is zero in every point not belonging to such a set is Riemann integrable, therefore (Lebesgue 1905, 216):

III of (Borel 1898). The preceding condition is also necessary; since a discontinuous function depends on a set more than enumerable of conditions, the power of the set of discontinuous functions is greater than the continuum. Note III of (Borel 1898) is interesting because for the first time a general theory of the functions is proposed and the two mathematical conceptions of the time, the idealist and empiristic mentalities exposed by Paul du Bois-Reymond in his *Théorie générale des fonctions* compete among them. Borel takes up a definite position against the idealist mentality. He adds in a note that this does not mean that it is impossible to find some result that can be applied to all functions like for example the beautiful theorem by Darboux about the definite integral: the superior integral and the inferior integral always exist. But results of such a kind are very unusual and the previous theorem itself, notwithstanding its theoretic beauty, cannot be applied if the function is too general.

[19] The proof is obscure, full of gaps and uses the second class ordinals. We will give it in Section 5 in the form proposed and completed by Lusin; we will also give an example of a simple function that is not Borel constructed by Lusin using the laws of arithmetic.
Borel (Borel,1922) considers again the question of the definition of a not analytically expressible function by means of transfinite ordinals in a very skeptical manner in the paragraph: *Sur les définitions analytiques et sur l'illusion du transfini.*
Lebesgue hints at measurable but not B measurable sets already in his thesis. In (Lebesgue 1904, 112) he writes in a note: "I do not know if it is possible to name a non-measurable function B; I do not know if there are non-measurable functions."

[20] It is well known that Lebesgue measurable sets that are not B measurable can be determined by the choice axiom.

[21] Obviously, it is a null Peano-Jordan measurable set.

It is possible to name Riemann integrable functions that are not B measurable.[22]

Lebesgue claims also that it should be interesting to begin a study of the functions and the sets that we can name and closes his analysis posing a question that has a particular meaning in the year in which Vitali published his example of a not Lebesgue measurable set using Zermelo's axiom (Vitali 1984, 231): *it is possible* to name *a not measurable set*? (As Lebesgue makes clear, the memoir was written in May 1904).

Analytic and Projective Sets by Lebesgue and Lusin

The whole memoir (Lebesgue 1905) was a source of ideas and methods that inspired many subsequent authors, as de la Vallée Poussin, Hausdorff, Souslin, Sierpinski, Kuratowski etc. It was also the starting point and furnished the matter of important researches made by Lusin; the work of this author consisted for many years of the development of some points of it; the outcome of this study was a deep treatment of the subject that can be found in many papers and in particular in the memoir (Lusin 1927) and the book (Lusin 1930).

Lusin was interested in comparing the concepts of a function defined by Borel by means of infinitely enumerable many conditions and a function that is possible to name in the meaning of Lebesgue: he thought that they coincide (Lusin 1930, 5) and to this end he submitted his forerunners work to a close analysis.

Indeed, after Baire classification of functions many other classifications of B measurable sets have been given: among them the first was furnished in (Lebesgue 1905). Precisely, at page 156, a B measurable set *E* is called by him *F* of α class if there exists a function *f* of α class such that *E*=$\{x: f(x) = 0\}$ and this is impossible by a function of less class than α. Analogously *E* is called *O* of α class if there exists a function *f* of α class such that *E*=$\{x: f(x) > 0\}$ and this is impossible by a function of less class than α. For α=0 the *F* sets are closed and the *O* sets are open. Obviously, the complement of an *F* set is an *O* set of the same class and vice versa. 182

Some years after in (de la Vallée Poussin 1915) we find another classification: a set is called to be of class α if its characteristic function belongs to the α Baire class. Lusin studied in detail all the properties of the Baire-de la Vallée Poussin classes of B measurable sets, but what is more interesting he studied deeply the projection operation in the class of B measurable sets. The starting point of this study was a mistake in (Lebesgue 1905, 191), where Lebesgue claims that *if a set E is B measurable then his projections are B measurable*. This, in general, is proved to be not true and Lusin started from another definition by Lebesgue to correct this statement. In (Lebesgue 1905, 165) Lebesgue observes that all the B measurable sets *E* can be described by means of analytic equalities or inequalities, that is there exists an analytically expressible function *f* such that *E*=$\{x: f(x) = 0\}$ or $\{x: f(x) \neq 0\}$; therefore they can be called *analytic sets*. The inverse proposition was never enunciated by Lebesgue and is not true. Then Souslin and Lusin, in order to complete Lebesgue's program producing the more general functions that can be named, used the new terminology for a class wider than the class of B measurable sets. They gave the name of *analytic sets* to all the sets that can be written by means of analytic equalities and only by them, i. e. they did not give the same name to their complements. In (Lusin 1930) the set *I* of irrational numbers of the interval

[22] All the subsets of a null Peano-Jordan measure set are Peano-Jordan null measure sets. Now the Cantor set C has null Peano Jordan measure and therefore all its subsets are null Peano Jordan measure sets: now since C has the power of continuum, its power set has the power greater than the continuum. But B measurable sets are numerable, thus there exist infinitely many null Peano Jordan measure sets that are not B measurable and therefore their characteristic functions are Riemann integrable but not B measurable. Lebesgue has the problem of *nommer* one of these functions.

(0,1) is considered and a set *E* is called *analytic* if there exists a continuous function *f(t)* on *I* such that *x=f(t)* for every $x \in E$. In (Lusin 1927) the two definitions are proved to be equivalent.

First Lebesgue had pointed out the theoretic interest of the *projection* as an operation that allows to form new sets it is possible to name starting from already known *sets*. Analytic and *projective* sets were studied by M. Suslin (1894-1919) and Lusin since 1917 and after in a long series of papers from 1925 to 1927 on the Comptes Rendus Acad. Sc., in (Lusin 1927) and the book (Lusin 1930); the theory of the projective sets was the subject of the lessons he gave at Moscow University in 1924-25. Sierpinski also was interested in projective sets in a series of papers.

The class of the *projective sets* contains all the analytic sets and their complements in the following way. Combining the operation of projection (P) with the operation of complement (C) it is possible to obtain, starting from B measurable sets E, all the analytic sets, as P*E*, then from their complements it is possible to obtain a new class of sets, C*PE*, that can be named even if their nature is unknown; again it is possible to obtain the new sets *PCPE* and so on: all these sets are the *projective sets,* a very difficult class to study. Lusin proves many properties of these sets: in particular, in (Lusin 1930, 144):

A set is analytic if and only if it is the orthogonal projection of a B measurable set;

and at page 152 and page 155:

The *analytic sets (and their complements obviously) are all measurable (but not necessarily B measurable).*

As a consequence:

The projective sets are not B measurable in general.

About Lebesgue's Construction of a not B Measurable Function which can be Named, as Exposed by Lusin

In (Lusin 1930) an interesting geometric interpretation of the example proposed in (Lebesgue 1905, 213-214) is given: Lusin dedicates a large part of his book in order to make it clear.

Lebesgue considers a sequence $z_1, z_2, \ldots$ whose terms are all the rational numbers of the interval *[0,1]* and a number *t* belonging to the interval *[0,1]* he writes in the form

$$t=\frac{\vartheta_1}{2}+\frac{\vartheta_2}{2^2}+\ldots,\ \vartheta_i=0,1$$

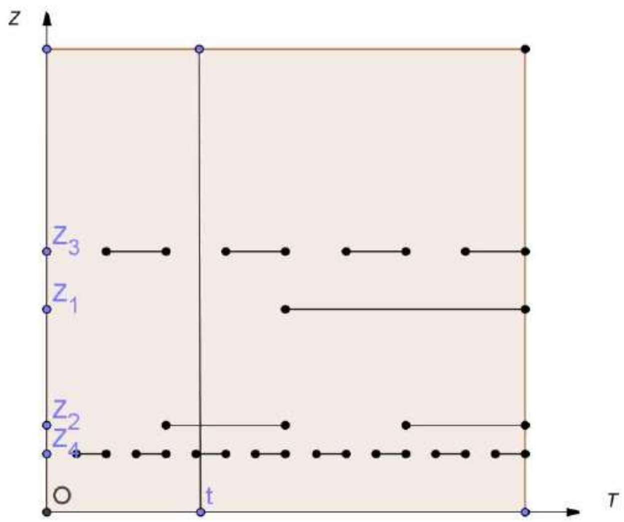
using only a finite number of digits *1* when this is possible. Now Lebesgue deletes in the sequence $z_1, z_2, \ldots$ all the z_i corresponding to indexes *i* such that $\vartheta_i=0$ and calls $z'_1, z'_2, \ldots$ the remaining *z*.

In (Lusin 1930, 198-202) this construction is interpreted in a geometric way that is much more clear and intuitive than the analytic one proposed by Lebesgue. Lusin considers in the *OTZ* plane the square $0\leq t\leq 1,\ 0\leq z\leq 1$. On the edge *t=0* he arranges all the elements of the sequence $z_1, z_2, \ldots$ For every natural *n* he considers the segment $z=z_n$ parallel to *OT* enclosed in the square; he divides such a segment in 2^n equal segments and considers only those of these segments, numbered in the increasing order of *T*, that are even. The so obtained figure is formed by infinitely enumerable many segments, parallel to *OT* axis: it is called by Lusin a *binary sieve* and denoted by Γ.

Obviously if $t=\frac{\vartheta_1}{2}+\frac{\vartheta_2}{2^2}+\ldots$ then $\vartheta_n=1$ if and only if the straight line D_t passing through t and parallel to OZ axis meets the straight line $z=z_n$ in a point belonging to Γ. Therefore, for every given t the preserved points z are the ordinates of the points in which the straight line D_t meets Γ.

Now, as Lebesgue says, given t, and therefore also the corresponding z', sometimes it is possible to find a symbol of α class in such a way that it is possible to establish a correspondence between the z' and the symbols $\beta<\alpha$ in such a way that to a z' corresponds only one β and vice versa and if $z'(\beta)$ and $z'(\beta_1)$ corresponds to β and β_1 then if $\beta<\beta_1$ it is $z'(\beta)<z'(\beta_1)$. Obviously, this is always possible if the number of the z' is finite. This means that there are some t such that the corresponding z' can be well ordered (in the increasing order of the OZ axis) and therefore an ordinal α can be associated with t. Let Ξ be the set of such t and let E be the complement of Ξ.. In (Lebesgue 1905) it has been previously proved that, given the ordinal α it is possible to name a function of class α, let it be φ_α. Then Lebesgue puts

$\varphi(t,x)=\varphi_\alpha(x)$ if t belongs to Ξ, $\varphi(t,x)=0$ if t belongs to E.



Lebesgue claims that this function is not B measurable (Lebesgue 1905, 214). In order to prove this, he proves that for every symbol α there exists a corresponding value t, and he claims that this is enough to prove that the function φ is not B measurable.

Lusin observes that this is equivalent to say that intersecting the binary sieve Γ with the straight lines parallel to OZ axis it is possible to obtain all the parts of the given sequence $z_1, z_2, \ldots$.

He makes clearer and completes Lebesgue's proof in his book: indeed in it he proves that by some preceding general theorems (Lusin 1930, 180 and 195) the set E is analytic and it is not B measurable if the sieve Γ is not bound, that is if there is no ordinal greater than all the ordinals involved in the sieve.

He obtains this result by an alternative more simple proof than Lebesgue: indeed, given an infinite part P of the sequence $z_1, z_2, \ldots$, he begins by showing that there exists one and only one point t such that the straight line D_t meets the binary sieve Γ in points whose ordinates are in P; to this end he considers the binary development $\frac{\vartheta_1}{2}+\frac{\vartheta_2}{2^2}+\ldots$. and puts $\vartheta_n=1$ if z_n belongs to P and $\vartheta_n=0$ otherwise.

The point $t_0=\frac{\vartheta_1}{2}+\frac{\vartheta_2}{2^2}+\ldots$ is the required point.

Now the set of the points $z_1, z_2, \ldots$ is dense in the interval *[0,1]* and therefore, by a Cantor's theorem, whatever the well ordered set *W* is, it is possible to determine a part of the sequence $z_1, z_2, \ldots$ that can be ordered in the increasing order in such a way to be equivalent to *W*. Then there is no ordinal greater than all the ordinals involved in the sieve.

Thus the set *E*, defined by Lebesgue and now exposed, is the first example in the literature of an analytic set (and therefore a Lebesgue measurable set) that is not B measurable. Lusin claims that the nature of the function φ(t,x) is unknown, but, obviously, the characteristic function of *E* is a Lebesgue not B measurable function obtained without using the choice axiom, but using all the transfinite ordinals of (0,1).

In order to avoid the recourse to the transfinite ordinals, in (Lusin 1927) Lusin *names* a not B measurable function in a completely arithmetic way, using Lebesgue's method for the proof and a completely arithmetic example given by Baire (Acta Mathematica, t. 30) of a function of the third Baire class. In fact, given the interval *(0,1)*, Baire considers an irrational number *x* written as a continuous fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

The function that associates every *x* with the sequence of natural numbers $a_1, a_2, a_3, \ldots$. is bijective. Baire puts $f(x)=1$ if such a sequence diverges, $f(x)=0$ otherwise and proves that the so obtained function belongs to the third class. Now Lusin considers an analogous function in the following way: a sequence $a_1, a_2, a_3, \ldots$.is called *compound* if among its terms there is a sequence such that its terms are divisible each by another, otherwise the sequence is called *simple*; then Lusin defines $f(x)=1$ if the sequence is compound, $f(x)=0$ otherwise. He uses Lebesgue's sieve to prove that *f(x)* is not B measurable.

His proof was proved to be wrong, but only in 1978 this simple function has been proved to be not B measurable by Kanovei (Kanovei 1978).

After Zermelo's Axiom

In 1904 Zermelo's axiom focuses the discussions on the ontological character of the mathematical entities: a new criterion is introduced in the discussion, and as time goes on, the mathematical positions about the problem are outlining and getting stronger.

In 1904 Baire, Borel, Lebesgue and Hadamard exchanged on this subject five letters, published in 1905 as *Correspondences in the Bulletin de la Société Mathématique de France* 33, 261-273; they were reprinted in (Borel 1914) under the title: *Cinq lettres sur la théorie des ensembles*.

In the first letter to Borel, Hadamard agrees with Zermelo but disagrees with Borel: Borel considers the choices of Zermelo's axiom equivalent to enumerate the elements of the set some after the others, this enumeration being transfinite. Hadamard observes that such an enumeration consists in the admission that the choice of one element depends on the others, while by Zermelo's axiom the choices are independent of each other. Moreover, there is no difference in the application of the axiom in the case of not enumerable infinitely many choices and in the case of only enumerable infinitely many choices. Finally, Zermelo does not give any manner to execute effectively some operation linked to the choice function and it seems doubtful that someone will be able in the future to give it. Certainly, it would be more interesting to solve the problem determining effectively the wanted correspondance, but as Borel poses the question, this would be not different from the problem of its existence: and this is the fundamental difference between a correspondence that can be defined and one that can be described.

On the other hand, in his letter to Hadamard, Baire denies the possibility of admitting the choice axiom since it is already senseless to consider the set of all subsets of a given set. "In particular, given a set . . . it is wrong for me to consider the parts of this set as data.[23] All the more, I refuse to attach any meaning to the fact of conceiving a choice made in every part of a whole."

In his letter to Borel Lebesgue makes a subtle difference between *nommer* and proving the existence. By the Cantor reasoning about the existence of an infinity not enumerable of numbers we are not allowed to name *une telle infinité*, but we cannot doubt about such an existence.

With the second letter to Borel, Hadamard distances definitively from him. He believes, like Cantor, that the set of all the subsets of the real interval (0,1) and the set of all the functions have a meaning and he understands that their power is greater than the power of continuum, but all this does not have any meaning for Borel, who requires the extra condition, without mathematical meaning, that all these entities have to be defined by a finite number of words. Two different conceptions of mathematics are facing each other and Hadamard does not intend to change his own. And finally: the law Lebesgue requires seems to be like the analytical expression that Riemann's opponents demanded with force.

Lebesgue will reply to this reproach maintaining a middle position between the platonic mentality of Hadamard and the constructive approach of Borel by the following words:

> In any question of existence, it is necessary to take into account two different mentalities: that of the idealist and that of the empiricist of P. du Bois-Reymond. In the foregoing, I have given the definitions their idealistic meaning [...] On the contrary, I made empiricist reasonings because they are the only ones that idealists and empiricists agree to declare correct.
> By giving definitions their idealistic meaning, we are sure to give them a meaning as broad as empiricists will never give them; but it does not follow that one must use idealistic demonstrations. (Lebesgue 1907, 211)[24]

In 1905 Borel answers Hadamard's letter putting an end to the correspondence, but also trying to reconcile the two different positions.

His attitude towards the choice axiom remains however inflexible: in 1914, in the second edition of his *Leçons*, p.161-162, he writes that the complete arithmetic notion of the continuum requires the possibility of enumerably infinite many choices. He believes such a possibility is very questionable, but anyway it is to be distinguished essentially from the possibility of not enumerably infinite many choices, that seems to him a complete nonsense.

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[23]Baire's position about the power set seems very similar to Borel's position; his mentality is very far from Lebesgue's theory as we can see in (Baire 1907), where he limits himself to classical measure and Riemann integral, without quoting Lebesgue's work.

[24] In 1910 Lebesgue will write: "It is necessary to add that various authors [VITALI, *Sul problema della misura dei gruppi di punti di una retta, Bologne,* 1905; LEBESGUE, *Sur les correspondences ponctuelles, etc. (Atti d. R. Acc. d. Sc. di Torino,* 1907*); Contribution à l'étude des correspondences de M. Zermelo (Bul. de la Soc. math. de France,* 1907*);* ED. VAN VLECK, *On non mesurable sets of points (Trans. of the Am. mat. Soc.,* 1908*)*] have indicated methods of forming non-measurable sets, but these methods suppose that operations are employed which are neither known nor possible to be characterized logically. ["Il est necéssaire d'ajouter que divers auteurs [VITALI, *Sul problema della misura dei gruppi di punti di una retta, Bologne,* 1905; LEBESGUE, *Sur les correspondences ponctuelles, etc. (Atti d. R. Acc. d. Sc. di Torino,* 1907*); Contribution à l'étude des correspondences de M. Zermelo (Bul. de la Soc. math. de France,* 1907*);* ED. VAN VLECK, *On non mesurable sets of points (Trans. of the Am. mat. Soc.,* 1908*)*] ont indiqué des procédés de formation d'ensembles non mesurables, mais ces procédés supposent qu'on emploie des opérations qu'on ne sait ni effectuer, ni même caractériser logiquement. (Lebesgue 1910, 371)].

When the choices are not enumerably infinite many we are not able to define the choice function, that is to distinguish it from another similarly generated function and therefore it is not possible to consider it as a mathematical being.

We finish this Section with the words of G. Pepe: he underlines in (Pepe 1984, 189) that at the beginning of the 20th century the disputes for priority of results or about foundations made more complicated the success of the researches about real analysis: the controversy between those who supported the use of Zermelo's axiom and those who rejected it was particularly severe. For example, in Italy, Vitali and Tonelli militated in two opposite alliances: Vitali accepted choice axiom and used it in its general form to prove the existence of a set not Lebesgue measurable, on the contrary Tonelli rejected it and elaborated an organization of variational calculus without it. Very probably for this kind of disputes the proof by Vitali of a not measurable set was published in a booklet, not inserted in a periodic review.[25]

The Controversy between Borel and Lebesgue

As the years went by, Borel exposed positions about the existence and the definitions of the mathematical entities more and more far from the ideas of Lebesgue. In (Borel 1912) he distanced himself from Lebesgue in an inflexible way, opening a controversy about the constructivistic and platonic approach to mathematical objects.

In the *Leçons* of 1898, Borel had given his definition of B measurable sets, by a construction from the bottom, starting from the real intervals and applying a finite or enumerable number of times the operations of union and complement.

In 1912 Borel completes his construction defining the integral for bounded analytically expressible functions as the limit of a sequence of integrals of polynomials that approximate the integrand function[26] and for unbounded functions as the limit of a sequence of Riemann integrals, extending a procedure already used by Jordan for functions with a finite number of singularities to the case of functions with enumerable many discontinuities.

In his exposition he claims priority of his ideas in introducing measure and superiority of methodological and mathematical character of his integral and, in general, of his conception of mathematical entities, defending Borel sets and functions (the only *bien-définis* sets and functions) against the apparent (in his opinion) generalizations introduced by Lebesgue.

For example, he says, even if Lebesgue has gone as far as "naming" a not analytically expressible function, it is necessary to point out that such a function is different from an analytically expressible one only on a null measure set and therefore the two functions are equivalent with respect to integration.[27]

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[25] As another consequence: "Carriere brillanti per Fubini e Tonelli, carriera stroncata per Vitali" (Vitali 1984, 12)

[26] Borel proves that every bounded analytically expressible function f is the limit in measure of a sequence of polynomials and defines the integral of f as the limit of the sequence of the integrals of such polynomials.

[27] And also, in the Note: "I leave apart the objections it is possible to make to the existence of the function named by Lebesgue. Lebesgue himself specifies the difference between naming and defining a function; but I would be more explicit than him and whenever all the transfinite numbers of the second class (and not only those less than a given one) appear, it would seem to me we are going out of mathematics." (Borel 1912, 208)
In 1898, Borel was already critical, but less drastic: "One can ask whether there is much illusion in the idea we have conceived, following G. Cantor, of the power of the second principle of formation." (Borel 1898, Note II)

Obviously, in this way, Borel keeps his distance also from Cantor's[28] mathematics; in the paper he confirms with force his aim: to distinguish between the calculations that can be really carried into effect, in order to give useful tools for applications and those for which this is not possible.

We find in this paper the definition of a calculable number and a calculable function for the first time: "We say that a number α is calculable if, for every given integer number *n*, it is possible to obtain a rational number whose difference from α is less than *1/n.*"[29]

All the rational numbers and all the irrational numbers for which an effective procedure for the calculus of their decimal digits is given are calculable.

This and the following definition of a calculable function represented the first attempts that will be followed by A. Turing and other mathematicians in the construction of recursive theory. For example, Borel considers a problem that is nowadays known as undecidable:[30] when two calculable numbers are equal? Obviously, the excluded middle principle cannot be used in this situation, since given two calculable numbers beside the possibility that they are equal or not there is also the possibility that we are not able to decide whether they are equal or not.

Here is the definition of a computable function:

> *We will say that a function is computable when its value is computable for any computable value of the variable*[31]

Borel claims that, since if α is a calculable number, it can be obtained with an arbitrary approximation, then also $f(\alpha)$ has to be calculated in the same way and therefore every computable function *f* has to be continuous.[32]

Borel knows the restrictions derived from his definitions but these restrictions are not arbitrary: they are necessary if we want to distinguish real mathematics from the purely verbal logical speculations, where the only concern is avoiding contradictions (Borel 1912, 166).

As it is evident Borel's pragmatic treatment is far from logic and Hilbert's formalist apparatus, and is closer to semi-intuitionist mathematics: his work is typically constructivist

[28] Fundamentally his work is developed inside Cantor's theory, with a fondness for enumerable sets and for sets generated by enumerable conditions and therefore having the power of continuum, feeling on the converse a strong dislike for the sets whose power is greater than the continuum. With respect to the continuum, since it cannot be omitted doing mathematics, Borel assumes sometimes a position similar to Kroneker's position with respect to the natural numbers, indeed he writes in 1898: "We will admit that the set *C* of the numbers between *0* and *1* is given, without searching how it could effectively be."

[29] Borel definition is very close to the definition of a computable real number: a real number *a* is computable if there exists a computable (by means of a Turing machine) function $f: N \rightarrow Q$ which for every natural number $n > 0$ gives a rational number *r* such that $|a-r|<1/n$.

[30] We can find the same type of problem already in (Borel 1898, Note II), where a procedure is described such that it is possible to recognize (in principle) if two continuous functions are different, but the procedure does not give an answer if they coincide.

[31] The definition seems quite simplistic: nowadays the mathematicians adopt a rather more complex definition for a computable function of a real variable.

[32] Borel dwells upon the continuity of calculable functions when he intends to calculate their approximate integrals: "An integral can be effectively calculable by means of the preceding methods if and only if the set of the points where the integrable function is not calculable has zero measure. This could be impossible if a function is assumed to be calculable only in the calculable values of the variables, since the set of these values has zero measure; but a function can be considered calculable in some not calculable values of the variables if it is known that in these values it coincides with a calculable continue function" (Borel 1912, 205-206).

and shares its philosophical approach with Heyting, who in those years agreed yet with Borel's ideas.

In (Lebesgue 1918) Lebesgue[33] answers quite irritated and puts for his part an end to the discussion answering all reproaches in detail for the last time. He defends himself against Borel's attacks about his methodological point of view, the mathematical definition of integral and priority questions.

He writes also that in his Memoire of the *Journal de Mathématique* in 1905 he was able with difficulty to name a not B measurable set and a not analytically expressible function and he thought it was impossible to encounter such a kind of sets or functions in analysis. He then agreed with Borel, who considered well defined only B measurable sets and functions. But now, in their paper on the *Comptes Rendus*, January 8, 1917, the Russian mathematicians Lusin and Souslin have announced they are able to name not B measurable sets and functions obtained by projecting B measurable sets in such a way that it is now a nonsense to search a general method that applies only to B measurable sets and functions (Lebesgue 1918,198).

Indeed if a set is Lebesgue measurable then *almost* all its projections are Lebesgue measurable and if a set is B measurable then all its projections are Lebesgue measurable (Fubini 1907), but they are not necessarily B-measurable; in general, as we have seen, they belong to an intermediate class between B-measurable sets and Lebesgue measurable sets, that is they are *analytic.*[34]

Borel answers immediately (Borel 1919) and makes clear another time his priority about the fundamental concept of measure, but he confirms also he holds Lebesgue's work in high esteem.[35] Moreover, about the result by Lusin and Souslin, he claims that if he had been informed when he wrote his memory, he should have changed something in it, but he adds that he would have treated the B measurable sets with a particular interest anyway, since almost all the sets a mathematician deals with are part of them. In the same manner, the discovery of the continuous functions without derivative does not stop the study of derivable functions (Borel 1919, 80).

With these words, the controversy can be said settled: Lebesgue will not mention the question anymore. In the note (1), p. 117 of the second edition of his *Leçons* he will observe that the principal advantage to study the measurable sets and not only the B measurable sets is not the fact that one can deal with a larger class of sets, but that one identifies the essential property of the sets to which it is possible to assign a measure, obtaining a self contained compatible theory. He adds that Borel seems to have considered in the class of the B measurable sets only those that can be obtained, starting from the real intervals, using the operations of union and complement a finite number of times, while, for a correct definition, also transfinite ordinals seem to be necessary.

But anyway, while Borel will maintain his position, Lebesgue, besides these technical observations, will be more conciliatory after some years and will affirm once more to be in debt with Borel for the concept of measure.

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[33] In that period Lebesgue was a candidate for a job as professor at the Sorbonne and was waiting his election to the *Académie des Sciences*.

[34] It is interesting to remember that Lusin, to underline the difficulty of the study of analytic sets, claims: "The author of this book inclines [...] to consider the examples constructed by himself as forms of words and not defining really accomplished beings, but only potentialities" (Lusin 1930, 322).

[35] About the constructive definitions he writes: "I do not see how it is possible to pretend to reason about a determined but not defined individual: there is a contradiction in terms about which I have often insisted; I believed so far that Hadamard was the only French geometer who did not agree with me on this point; but I am realizing that Lebesgue's ideas have undergone a change in the opposite direction to mine and while in 1905 he roughly agreed with me against Hadamard, he does not agree with me today; but I would not go so far as to say that he agrees with Hadamard" (Borel 1919, 76-77).

Lebesgue's Contribution to Variatonal Calculus.

We conclude this paper with a contribution of Lebesgue to Variational Calculus as is exposed in a manuscript found in a drawer of his office and published after his death (Lebesgue, 1963).

As we have already seen, in the first years of the 20th century the mathematicians did not think highly of the researches about real analysis, except for the paper about trigonometric series, because they had no applications. Lebesgue was aware of this fact as he acknowledges sometimes. So besides the fundamental contribution he gave to the theory of trigonometric series, we have to refer to the notes he wrote about Variational Calculus, an important topic where the new theories revealed their power. These notes were prepared for a course he gave at the French College, where he exposed in a clear manner the classical and the direct method.

Lebesgue begins observing that the direct method of the Variational Calculus corresponds exactly to the Weierstrass theorem about the existence of the minimum and the maximum of a continuous function of one or more real variables: given a functional $F(P)$, where P is a function of one or more real variables (Hadamard first used the word *fonctionelle*), given a minimizing sequence of functions, $P_1, P_2, \ldots$ i.e. a sequence such that the sequence $F(P_1), F(P_2), \ldots.$ tends to the glb of the values of F, if the function P_o is the limit of the sequence $P_1, P_2, \ldots$ and if F is continuous in a suitable sense, then the value $F(P_o)$ coincides with the glb, that is, $F(P_o)$ is the minimum of the functional. Lebesgue claims that this method was introduced by Arzelà in a very important paper that passed unnoticed, so Hilbert exposed it at the Zurich Congress in 1897 reproducing in some way the paper he did not know. Only after few years, Arzelà decided to point out his priority and therefore the direct method on Variational Calculus has to be ascribed to both Hilbert and Arzelà. But there was a considerable counterpart to the papers of Hilbert and many mathematicians dealt with this new procedure after him. It consists in considering a minimizing sequence $P_1, P_2, \ldots.$: it always exists; then a limit element P_o has to be considered, that does not exist in general. A paper by Ascoli, (Ascoli,1883), furnishes the following result,[36] the analogous of Bolzano Weierstrass theorem for the functions:

if the functions of a sequence are *uniformly bounded* and *equicontinuous*, that is for every $\varepsilon>0$ there exists $\sigma>0$ such that $|f_i(t+h)-f_i(t)|<\varepsilon$ for every i and $|h|<\sigma$, then the sequence has a limit and the limit function is continuous too. If F has some continuity property, then $F(P_o)$ is the required minimum. Now, in general, the functionals are continuous only in particular cases. Lebesgue remembers that in his thesis he had extended the definition of lower or upper semicontinuous function in a point given by Baire to the functionals: then he was inspired just by a philosophical interest, but as it often happens, his idea turned out to be very useful for the direct method. A functional $F(P)$ is called *lower semicontinuous* in P_o if the smallest of its limits when P tends to P_o is at most equal to $F(P_o)$ (in an analogous way an *upper semicontinuous* functional can be defined). It is now clear that if a functional F is lower semicontinuous and if P_o is the limit obtained before then $F(P_o)$ is the minimum. Lebesgue remembers also that Leonida Tonelli proved that in all the problems called *regular* the lower continuity comes true and therefore it is possible to apply the direct method as before. It is for this reason that Lebesgue underlines the minor part he played in this progress of the method, at the moment that many authors ascribe the idea of semicontinuity for functionals to Tonelli.[37]

[36] Ascoli presented this paper to the competition for the Royal Prize for the Mathematics in 1883, announced by the Accademia dei Lincei. No competitor won the prize because the board of examiners did not judge the produced papers enough complete and postponed the competition until two years after.

[37] See (Tonelli 1915) and (Tonelli 1920).

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