

Article

Contribution of Italian Mathematicians to Real Analysis in the last Decades of Nineteenth Century

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Abstract:

In (Biacino 2018) the evolution of the concept of real function of a real variable at the beginning of 1900 is outlined, reporting the discussions and the polemics, in which some young French mathematicians of those years as Baire, Borel and Lebesgue were involved, about what had to be considered a genuine real function. In this paper, I consider in particular the contribution to real analysis theory done by some Italian mathematicians as Volterra, Peano, Ascoli, Arzelà, etc., in the last decades of nineteenth century before the introduction of measure and integration theory by Lebesgue.

Keywords: Integrable functions in Riemann's sense; Nowhere dense subsets; Outer content; Peano-Jordan measure; Reduction of double integrals; Term by term integration

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Introduction

One of the most interesting problems many mathematicians were faced in the second half of nineteenth century can be outlined as follows: there are two ways to define the integral, in order to calculate an area. One way gives the definite integral and was introduced first by Mengoli, found again by Cauchy and perfected by Riemann. Riemann had met in Berlin the Italian mathematicians Betti, Brioschi and Casorati in 1858 and had established friendly relation with Betti and Beltrami, at that time professors at Pisa, living for one year in that town in 1864. In order to calculate the area of a trapezoidal figure that is a rectangular trapeze whose oblique edge is the diagram of a bounded function $f(x)$ with respect to the x -axis, Riemann substituted this figure by another one where the curve is a scale whose steps are segments parallel to the x -axis. The area of this last figure is then given by $\sum_{i=0}^{n-1} (x_{i+1} - x_i) f(z_i)$, where f is defined in the interval $[a, b]$, $a = x_0 < x_1 < \dots < x_n = b$ and, for every $i = 0, 1, \dots, n$, the point z_i belongs to $[x_i, x_{i+1}]$. If the limit of this sum exists when the maximum width of the intervals tends to zero and it is finite, then it is the *definite integral*. In his famous dissertation of 1854, published only in 1867, Riemann called the functions for which the operation of definite integral has a meaning as *integrable functions*.

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Another way, the classical way to define area, consisted in considering integration as the inverse operation of derivation, that is, if a function $f(x)$ is given in an interval $[a,b]$, we can consider a new function $F(x)$ whose derivative in every point is $f(x)$, called a *primitive* of $f(x)$.

Then the integral is given by definition by: $\int_a^b f(x)dx = F(b)-F(a)$.

If a function has a primitive does it have also a definite integral in Riemann sense? For very regular functions it is so. But the doubt that the two properties were not equivalent for general functions was posed already by Dini.

U. Dini (1845-1918) was a disciple of Betti interested at first in differential geometry; permeated by the new ideas, he was able in few years to give very important contributions to integration theory and real analysis and to create an important mathematical school operating in this field; it is why the theses of Baire, in 1899, and Lebesgue in 1902 were published on the *Annali di Matematica*, directed by Dini. Indeed, in 1878, the important Italian mathematician had already developed in his famous treatise *Fondamenti per la teorica delle funzioni di variabili reali* significant ideas in the study of the functions of a real variable in a very general context; there he gives the notion of indetermination limits in a point, on the left and on the right, for oscillating functions and the consequent definition of derivative numbers, the proof of the Hankel condensation principle and so on.

As we will see, some followers of his school continued his researches about real analysis giving many interesting contributions. For example, Rodolfo Bettazzi (1861-1941) in 1884 postulated, in order to define the integral of a function of more variables in a very general setting, some property of measurability of the domain of integration before the papers by Peano and Jordan had come to light (Bettazzi 1884). In the same years also the German mathematician C. G. A. Harnack (1851-1888) was working about this topic, assuming in his book about derivation and integration that the boundary curve of a plane domain was such that it could be enclosed in a plane domain whose magnitude was arbitrarily small, the same holding also for the curves by which the region was portioned (Hawkins 2002). We are not surprised at that, since, as we have already said, the work of some Italian mathematicians is strictly linked in the last decades of nineteenth century to the investigations born in Germany with the development of Riemann theory of integration, as the problem of characterization of the Riemann integrable functions, or the generalization of some properties of Cauchy integral to Riemann integrable functions such as integration by parts or the extension of the integral to functions of more real variables, and the problem of the reduction of multiple integrals, etc. In this period the most important exponent of the German school of Weierstrass was Hermann Hankel (1839-1873). Like his master Riemann and Dirichlet, he conceived functions as very general entities. In order to characterize Riemann integrable functions in 1870, in a long essay, *Untersuchungen über die unendlich oszillierenden und unstetigen Funktionen*, he had introduced the local concept of *jump* (a forerunner of the concept of oscillation in a point) in the following way: given $\sigma > 0$ a function f is said to make *jumps greater than σ* in a point x if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\delta| < \varepsilon$ and $|f(x+\delta)-f(x)| > \sigma$. The *jump in the point x* is then defined as the least upper bound of the set of σ such that f makes jumps greater than σ in x (Dini will call it *salto*). Then he called a function a *pointwise discontinuous function* (*punktiert unstetige Funktionen*), if, for every σ the set S_σ of the points x such that the jump of f in x is greater than σ is *nowhere dense*, that is it is a set whose closure has no interior points. His most important theorem claims that

A function is Riemann integrable if and only if it is a pointwise discontinuous function.

As we will see this condition is only necessary but this is the first time that an integrable function is characterized by means of the set of its discontinuity points.

Among other things Hankel proves that:

If f is pointwise discontinuous in (a,b) then its points of continuity constitute a dense subset of (a,b) .

Indeed let f be pointwise discontinuous: then for every $\sigma > 0$ and for every interval $I \subseteq (a,b)$ there exists an interval $I_1 \subseteq I$ that is free from points of S_σ . Hankel iterates the previous consideration: there exists an interval $I_2 \subseteq I_1$ that is free from points of $S_{\frac{\sigma}{2}}$ and so on for every $n \in \mathbb{N}$ there exists $I_{n+1} \subseteq I_n$ that is free from points of $S_{\frac{\sigma}{2^n}}$. Hankel implicitly assumes the intersection $\bigcap I_n$ is nonempty and claims that it contains points of continuity of f . Thus the theorem is proven. This proof is given also in (Dini 1878, 63), where at every step the width of I_{n+1} is less than the half of the width of I_n , in such a way that the limit of the increasing sequence of the inferior extremes and that of the decreasing sequence of the superior extremes of the intervals coincide and the common limit is a point of continuity.

In what way did Hankel attain his most important theorem? He considered in his essay some singular functions whose discontinuities did not fill any interval: their set could be enclosed in intervals of arbitrarily small total magnitude. Such functions perhaps suggested him that a function is pointwise discontinuous if and only if its discontinuities form a set that can be enclosed in infinitely many intervals whose total length could be made arbitrarily small.

Now such a set is an enumerable union of nowhere dense sets, but the converse does not hold. Then the proof of the main theorem is erroneous. Indeed, Hankel mistook nowhere dense subsets (whose outer content may be also positive) with *first species sets*, that is with sets that have a derivate set of finite order equal to the empty set: the outer content of these latter sets is obviously zero. But we have to notice that the idea of a measure of a set was foreign to Hankel's thinking.

The Irish mathematician H. J. Smith since 1875 proved that there are nowhere dense sets (nowadays known as generalized Cantor sets), whose outer content is positive, but his work passed unnoticed in the continent. And in those years many authors often confused the concepts of nowhere dense subsets, first species sets and sets that can be covered by a finite number of intervals whose total sum can be made arbitrarily small (the so called negligible sets), as we can read for example in (Letta 1994) or (Hawkins 2002).

Nowhere Dense Sets with Positive Content and Derivatives that are not Integrable

With regard to Hankel theory of pointwise discontinuous functions Dini pointed out that if a function f is Riemann integrable then it is pointwise discontinuous (Dini 1878, 250) that is it has infinitely many discontinuities but in every subinterval of the interval where it is defined there is at least a point of continuity for f (Dini 1878,62); but he claimed also that he believed that the converse in general does not hold. He was not able to construct an example showing that there exist nowhere dense sets whose content is positive to prove the partial fallacy of Hankel proof. However, he proved that every pointwise discontinuous function obtained by the principle of condensation of singularities is integrable, and in general:

Given a function defined in the interval (a,b) , if it possesses only first species discontinuities, or, if it possesses second species discontinuities, these form a set of first species then the function is integrable. (Dini 1878, 246)

The proof is obtained using the fact that in the present hypotheses the set of the discontinuities of second species can be enclosed in a finite number of intervals whose length can be made as small as we want. Observe that it is evident from this proposition that it was not clear yet that measure theoretic concepts, instead of topological ones, had to be

introduced in order to characterize the set of the discontinuity points of an integrable function.

In 1881, an example of a nowhere dense set that is not negligible was furnished by a disciple of Dini, a twenty aged student at the *Scuola Normale* of Pisa, Vito Volterra (1860-1940) (Volterra 1881a). In the same year, Volterra proposed also the first example of a derivative function that is not Riemann integrable, but pointwise discontinuous (Volterra 1881b).

He begins the first paper proving that given two pointwise discontinuous functions defined in the same interval there exist in every subinterval points where the functions are both continuous. For the proof he borrows from Hankel the method used to prove that the points of continuity of every pointwise discontinuous function are dense. Therefore, the sum of two pointwise discontinuous functions is pointwise discontinuous;² as a consequence he obtains a generalization of the following theorem by Hankel and Dini:

If a function is discontinuous in every irrational point of a given interval then it is totally discontinuous. (Dini 1878, 63)

At this point, Volterra gives his famous counterexample of a nowhere dense set whose outer content is positive. He exhibits a function (Volterra 1881b) with bounded derivative which is not integrable. We give here his proof revisited by Lebesgue (Lebesgue 1904a, 43).

Consider a sequence of real numbers t_n such that the infinite product $T = \prod t_n$ is convergent (not null); then from the middle part of the real interval $(0,1)$ remove the open interval of length $1-t_1$, obtaining two intervals each of length $t_1/2$, from the middle parts of the two remaining parts remove two intervals as before with t_2 in place of t_1 , each one having length $t_1/2(1-t_2)$, obtaining four intervals each of length $t_1 t_2/4$ and so on. Let R be the generalized Cantor set obtained after all these operations are performed. Observe that if $t_n=2/3$ we obtain the Cantor set, but in this case $T=0$; for our purposes t_n could be for example equal to $\frac{4n^2-1}{4n^2}$. Now R has outer measure equal to T , while the inner measure is 0. So it is a closed, nowhere dense, not measurable set. Consider the function $f(x)$ in the interval $(0,1)$ that is zero in all the points of R and is defined in the following way in every removed interval (a,b) :

$$f(a)=0, f(x)=(x-a)^2 \operatorname{sen} \frac{1}{x-a} \text{ if } a < x < a+c, f(x)=c^2 \operatorname{sen} \frac{1}{c} \text{ if } a+c \leq x \leq b-c, f(x)=-(x-b)^2 \operatorname{sen} \frac{1}{x-b} \text{ if } b-c < x < b, f(b)=0,$$

where c is a suitable number $0 < c < (b-a)/2$.

The function $f(x)$ is continuous, obviously it is derivable in the points not belonging to R . If $x_0 \in R$ it coincides with the extreme of a removed interval (a,b) and therefore $|\frac{f(x_0+h)-f(x_0)}{h}| = |\frac{f(x_0+h)}{h}| \leq |h|$ whence $f'(x_0)=0$. So $f(x)$ is derivable everywhere.

The derivative $f'(x) = 2(x-a) \operatorname{sen} \frac{1}{(x-a)} - \cos \frac{1}{(x-a)}$ for $a < x < a+c$, that is near a point a of a removed interval (a,b) , is bounded and the maximum and minimum limit for $x \rightarrow a$ are 1 and -1; therefore $f'(x)$ is discontinuous at all the points that are extremes of removed intervals, that is at the points of R , whence it is pointwise discontinuous: since the measure R is the number $T > 0$ it follows that $f'(x)$ is not Riemann integrable.

But, after the appearing of Lebesgue's theory of integration, it will be clear that Volterra's derivative function is Lebesgue integrable, since, by Baire (1899, 64), every

² In (Baire 1899, 66) and (Lebesgue 1904b), Volterra's proposition is generalized for the case of infinite enumerable functions.

derivative is measurable, and a bounded measurable function is Lebesgue integrable. This function is a pointwise discontinuous function, but it is not Riemann integrable.

The previous example by Volterra proves that Dini was right: pointwise discontinuous functions exist whose set of discontinuities has positive outer content. Mainly it proved the inadequacy of Riemann integral to solve the problem of the research of the primitives of a given bounded function, the problem we start with, and justified the introduction of Lebesgue integral, as Lebesgue himself notices in (Lebesgue, 1904a).

Some years after, L. Tonelli in an early paper of his (Tonelli 1907) determined a necessary and sufficient condition in order a derivative function in an interval (a,b) is Riemann integrable.³ He proved also that:

If a derivative is a function of bounded variation then it is continuous, and a function of bounded variation is a derivative if and only if it is continuous.

The young Tonelli in 1908 gave another complement to Dini's study about the singularities of a function. He proved that a function of a real variable has at most enumerable many 1st species discontinuities (Tonelli 1908). Therefore, every function without 2nd species discontinuities, by the result by Dini quoted before, is Riemann integrable. Moreover, such a function is pointwise discontinuous on every perfect set and therefore representable as the sum of a polynomials series, uniformly continuous on every closed interval contained in an open continuity interval, if such intervals do exist (and Riemann function proves that may be that such an interval could even do not exist). This was in accordance with an early work by C. Severini written in 1897 where the Weierstrass approximation theorem is extended to a class of integrable functions.

A New Characterization of Integrable Functions by Ascoli and du Bois-Reymond: The Definition of Outer Content

Let us come back to the problem of the characterization of an integrable function. An important contribution was given by the Italian mathematician G. Ascoli (1843-1896) in (Ascoli 1875). He defines the *oscillation* of a real function $f(x)$ in a point y as the difference between the maximum and the minimum limit of $f(x)$ in y . Moreover, on the basis of such a punctual definition of oscillation, he furnishes the following condition (for the proof see (Ascoli 1875) and (Letta 1994)):

A function is integrable in (a,b) if and only if for every $\varepsilon > 0$ and for every sequence of decompositions of (a,b) , G_1, G_2, G_3, \dots in partial intervals such that the maximum width of the intervals of the decomposition G_n tends to 0, the sum of the measures of the intervals containing points where the oscillation is equal or greater than ε tends to 0 when n tends to ∞ .

The preceding condition is in general referred to as a criterion established in 1882 by du Bois-Reymond, a disciple of Weierstrass at Berlin and one of the most enthusiastic

³ Tonelli in this paper considers an infinitesimal sequence $h_1, h_2, \dots, h_n, \dots$ of real numbers and the corresponding sequence of functions:

$$\frac{f(x+h_1)-f(x)}{h_1}, \frac{f(x+h_2)-f(x)}{h_2}, \dots, \frac{f(x+h_n)-f(x)}{h_n}, \dots$$

Then $f'(x)$ is Riemann integrable if and only if there exists $L > 0$ such that $|\frac{f(x+h_n)-f(x)}{h_n}| < L$ for every $n \in \mathbb{N}$ and for every $x \in (a,b) - F$, where F is at most an enumerable set, and the sequence is almost uniformly convergent in the sense of Arzelà.

exponents of Riemann's theory of integration, familiar in the early 1870 with the work of Riemann, Hankel, Heine and Cantor. Du Bois Reymond considered "assumptionless functions" that is functions in their most large sense, defined as whatever correspondence between sets in the sense of Dirichlet, whose only restriction was to be Riemann integrable (Bottazzini 2003). Here is the criterion established by him:

A real bounded function $f(x)$ in the interval (a,b) is integrable if and only if for every $\varepsilon > 0$ the set of the points where the oscillation is greater than ε is an integrable set.

In the terminology of du Bois-Reymond an *integrable set* is just a set that can be covered by a finite number of intervals the sum of whose lengths is less than σ . Both Ascoli and du Bois-Reymond considers the points where the oscillation is greater than ε . Du Bois-Reymond was the first to define the indetermination limits in 1870 and consequently the oscillation as the difference between them (Hawkins 2002).

The paper by Ascoli is written in Italian and is difficult to read, perhaps this is why it passed unnoticed. Lebesgue and also the Italian Vitali will refer always in their works to du Bois-Reymond criterion and will not quote it. It is also worth noticing that in the same year in which the *Memoir on discontinuous functions* by Darboux appears, the existence of upper and lower integrals of a bounded function is proved in (Ascoli 1875).

The studies about the integrable functions stimulated researches about the kind of the set of their discontinuities and in few years, it became clear that not topological but measurability conditions were adequate in order to try to solve the problem. In 1884 the Austrian professor at the University of Innsbruck Otto Stolz (1842-1905) was the first to introduce in a paper on *Math. Ann.*, the journal directed by Klein, the definition of the *outer content* of a set E enclosed in an interval (a,b) as a number L such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite partition P of (a,b) in intervals with maximum length of the intervals less than δ it is $|L(P)-L| < \varepsilon$, where $L(P)$ is the total magnitude of the intervals of P that cover E .

Some months after, in 1884, Cantor gave a different but equivalent definition of content. In 1885, also Harnack gave independently a definition very similar to Stolz's one: he pointed out that the starting point had been the fact that some sets can be covered by a finite number of intervals with arbitrarily small total length: Harnack called them *discrete* since they have a behavior similar to finite sets. He also considered countable sets; it may be that the example given by Hankel of the set whose elements are the infinitely many points $\frac{1}{2^n}$ suggested to him that countable sets also could be enclosed in intervals such that the sum of their lengths is as small as we want. Indeed if a_1, a_2, \dots are the elements of a countable set E , given $\delta > 0$, it is possible to enclose every element a_n in an interval whose length is ε_n in such a way that $\sum \varepsilon_n < \delta$. Perhaps these considerations suggested to Borel some years after the introduction of Borel zero measure sets to define his measure theory. But Harnack did not introduce a measure for countable everywhere dense sets. Perhaps this is because, while he was well informed that nowhere dense sets could have not negligible measure, he believed however that countable everywhere dense sets were too large from a topological point of view to be considered negligible. Hawkins thinks that the fact now so familiar to appear commonplace that certain everywhere dense sets can be enclosed in intervals of arbitrarily small total length must have seemed too paradoxical to serve as the basis for a theory of measure (Hawkins 2002,64).

For many mathematicians the definition of content was not related to the concept of integral: this relationship became clear with the work of Giuseppe Peano (1858-1932). Since the notion of area for a region with a curvilinear boundary had not yet received a precise and suitable meaning, Peano thought that it was necessary to perform a complete theory for that (Peano 1883). For a bounded region E in the plane he considered two classes of polygons: those entirely enclosed in E and those including E . The areas of the polygons of the first class

have a least upper bound, the areas of the polygons of the second class have a greatest lower bound. If these two numbers coincide then their common value is by definition, the area of E , but if they differ the concept of area does not apply. It is clear by his own words that Peano was led to the previous definitions by the analogy with the definition of the integral and by the specification of the definition of l.u.b. and g.l.b. These ideas were generalized and exposed for sets of one, two or three dimensions in Chapter V, *Geometrical Magnitudes*, of (Peano 1887).

Here Peano gives rigorously the definition of *interior*, *exterior* and *limit point* (nowadays boundary point) for a given set; he considers as an example the linear set A whose abscissas are rational numbers greater than 0 and less than 1. This set has no interior points but all the points of the closed interval are boundary points, while all the other points are exterior.

Perhaps motivated by Volterra which had used the names of lower integral and upper integral (Volterra 1881b), Peano introduces the concept of internal and external length for a linear set E respectively as the l.u.b. of the lengths of the finite unions of the intervals contained in E and the g.l.b of the lengths of the finite unions of intervals containing E . In the case of the set A whose abscissas are rational numbers greater than 0 and less than 1 he observes that the internal length of A is 0 while the exterior length is 1 and therefore the set cannot be given a length. If from a set formed by a finite number of segments and containing a set A we take away a set formed by a finite number of segments contained in A , we obtain a finite number of intervals whose union contains the boundary of A . Peano observes that A has length if and only if the sum of the widths of these intervals is as small as we want, that is:

A set A is measurable if and only if its boundary has null measure.

In every case the difference between external length and internal length of a linear set is the external length of the boundary.

Similar definitions and theorems Peano gives for the area and volume of the plane and tridimensional sets. Peano was probably led to such a general approach by the reading of the papers on *Acta Mathematica* 1883-84 that Georg Cantor (1845-1918) dedicated to the infinite and linear sets of points and the general notion of volume of whatever n -dimensional set. But Cantor measure is an outer content and therefore it is not additive.

In (Peano 1887) there is a strict connection between measure theory and integral theory: if the function f is nonnegative and E denotes the figure under its graph, then the lower integral coincides with the internal area of E and the upper integral coincides with the external area of E , in such a way that f is integrable if and only if E is measurable.

Five years after the concept of measurability will begin central in the theory of integration for the treatment given by Jordan in the second edition of his *Cours d'analyse* (Jordan 1893), where it will assume a determinant role in the definition of multiple integrals.

Other Contributions to Real Analysis by Peano: Another Definition of Limit

It is interesting the contribution Peano gave in the first years of his activity to (Genocchi 1884), an internationally known treatise, that was translated even in Russian. In April 1882 Angelo Genocchi (1817-1889), which was teaching infinitesimal analysis at the University of Turin, had fractured one of his knees. In that period the young Peano (1858-1932) was his assistant (until 1890 when begun teaching as an *extraordinaire* professor, full professor from 1895) and substituted his professor during the time he was ill, that is from May 1882 until March 1884). One of the first lessons Peano gave (May 1882) was about the area of a surface. At that time the precise definition was not yet enough clear even if Lagrange had calculated

such an area in 1756 by a double integral. Serret in his *Cours de calcul différentiel et integral* had given a definition similar to that of the length of a curve, calling area of a surface S limited by a curve g the limit of the area of a polyhedral surface, whose faces were triangles, inscribed in the surface and bounded by a polygonal line tending to g . To make this definition sound it was necessary to prove that the limit exists and is independent from the rule by which the triangular faces decrease. Peano found a counter example proving that the previous theorem is wrong and sent it to Genocchi, being in the dark about the fact that in December 1880 also Schwarz had sent two letters to Genocchi for the same reason, enclosing a paper model, and in January 1881 he had sent also a description of his counter example. Ugo Cassina exposed this singular fact of the almost contemporaneous discovery of the counter example in (Cassina 1950), where the correspondence of Genocchi with Schwarz and Hermite about this topic is published. The example was published only after some years by Schwarz in the second volume of his *Gesammelte Mathematische Abhandlungen* in 1890; also Peano inserted several years after an abstract with his counter example in the Tome IV of his *Formulaire mathématique* (1901-03). This episode (Borgato 1991b) shows how teaching and researching were linked in Peano's activity and how a great maker of counter examples he was.

After he had substituted Genocchi, Peano thought to publish Genocchi's lessons supplemented with his own explications (Genocchi 1884): in particular for this book he wrote the *Preface* and the *Notes (Annotazioni)* where he gives for every paragraph a great deal of historical and bibliographical notes about the theorems and their exposition in preceding treatises, quoting almost all the bibliography that in those years was available with regard to real analysis, from Cauchy, Weierstrass, Heine, Darboux to Dini, Dedekind, Cantor, Harnack, du Bois-Reymond, Stolz, Jordan, etc. Peano gives a fundamental importance to the functional point of view, so draws attention to many imprecisions and mistakes some of which were reproduced in the texts even after the fact that they were erroneous had been proven: such as the incorrect definition of continuity, given in some treatises and already underlined in (Darboux 1875); the geometrical proof of the zeros theorem for continuous functions by Cauchy: for it Peano observes only an algebraic proof of the theorem allows the geometrical intuitive considerations. Moreover he underlines the wrong enunciation, given in some treatises, of the Theorem of de L'Hospital, or the theorem about the continuity of a function of two variables partially continuous with respect to both variables, or about the derivability of a compound function of more variables whose partial derivatives are not continuous etc. Peano observes that, in order to formulate Lagrange's theorem about finite differences, it is not necessary to add the condition that the derivative is continuous as Jordan had done, and he adds a famous generalization to three functions of it; gives the example of a function whose second derivatives do not commute, and many other examples and counterexamples.

There are also many other interesting observations, due to the original Peano's way to do mathematics: for example, in *Annotazione N.7* he gives another formulation of the definition of limit, that is equivalent to the classical one, but differs from it. Peano considers for simplicity the case that the independent variable x tends to infinity and says that a function f is *greater (respectively smaller) than a number a when x increases indefinitely* if for x greater than a certain value it is $f(x) > a$ (respectively $f(x) < a$). Peano then divides the numbers in three categories:

- 1) the numbers a such that $f(x)$ is greater than a when x increases indefinitely;
- 2) the numbers a such that $f(x)$ is smaller than a when x increases indefinitely;
- 3) the remaining numbers.

He then proves the Theorem:

The function $f(x)$ tends to a limit a if and only if there exist numbers of the first and the second category but there is only one number of the third category or there is none.

In other words, if we denote by A and B the sets of numbers of first and second category respectively, then A and B are two unbounded intervals, on the lower side the former, on the upper side the latter, and are separated. Then $f(x)$ is convergent if and only if A and B are contiguous and in this case the limit coincides with the unique separation element. If $A = \emptyset$ and $B = \mathbb{R}$ ($A = \mathbb{R}$ and $B = \emptyset$) then $f(x)$ is negatively (positively) convergent. This is Peano's reasoning strictly linked to the fact that real number theory had been introduced only twelve years before.

Simplifying a Geometrical Definition of a Derivative.

It seems so quite natural Peano's interest in the definition of fluxion given in (Maclaurin 1742, 579), where general rigorous geometrical principles that seem immediately applicable to algebraic quantities are used, without considering the fluxion generated by the motion or considering the first or last ratios of evanescent increments, but using algebraic inequalities:⁴ we refer the translation of them in few words as Peano does in his *Annotazione N.32*. A fluxion of a quantity with respect to another quantity is

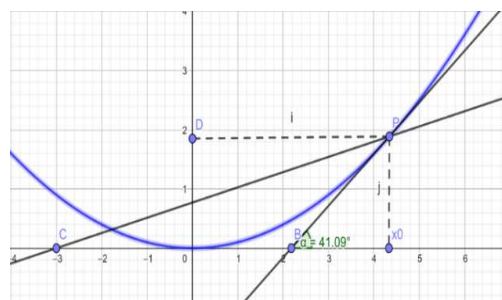
Any measures of their respective rates of increase or decrease while they vary, or flow, together.

Maclaurin uses the expression:

A quantity increases by differences that are always greater than the successive differences by which another quantity increases.

Therefore if $f(x)$ and $g(x)$ denote quantities, if $f(x_0) = g(x_0)$ and $h > 0$, then from

$$f(x_0+h) - f(x_0) > g(x_0+h) - g(x_0) \text{ and } f(x_0) - f(x_0-h) > g(x_0) - g(x_0-h)$$



it follows $f(x_0-h) - g(x_0-h) < 0$ and $f(x_0+h) - g(x_0+h) > 0$; Maclaurin makes also clear that small values of h are to be considered. Precisely, Peano translates the preceding words saying that a function $f(x)$ increases more rapidly than a function $g(x)$ for $x = x_0$, if $f(x) - g(x)$ is (strictly) increasing in $x = x_0$. In this case Peano says also that $g(x)$ increases less rapidly than $f(x)$. For a

⁴ This way as a basis for the definition of limit can be found in d'Alembert, L'Huilier, Lacroix, that all quote Maclaurin.

linear function the ratio of the increments of the dependent and independent variables is constant and Maclaurin gives such a constant the name of derivative of the linear function in every point. For a not constant function $f(x)$ Peano translates Maclaurin's words writing that it has derivative equal to $f'(x_0)$ if:

For $x=x_0$, $f(x)$ increases less rapidly than every linear function having a derivative greater than $f'(x_0)$ and more rapidly than every linear function having a derivative lower than $f'(x_0)$.

We can conclude Peano's reasoning saying that $f'(x_0)$ is a number (if it exists) such that:

for every $k > f'(x_0)$ there exists a neighborhood I of x_0 such that:

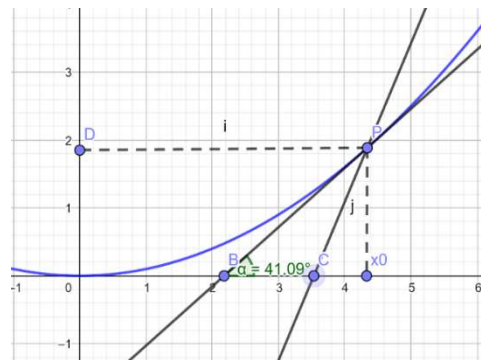
$$(x, y \in I, x \neq y, x \leq x_0 \leq y) \Rightarrow k > \frac{f(x) - f(y)}{x - y};$$

for every $k < f'(x_0)$ there exists a neighborhood I of x_0 such that:

$$(x, y \in I, x \neq y, x \leq x_0 \leq y) \Rightarrow k < \frac{f(x) - f(y)}{x - y}.$$

Therefore $f'(x_0)$ is a number (if it exists) such that for every $\varepsilon > 0$ there exists a neighborhood I of x_0 such that:

$$x \in I, x \neq x_0 \Rightarrow f'(x_0) - \varepsilon < \frac{f(x) - f(x_0)}{x - x_0} < f'(x_0) + \varepsilon.$$



In this way Peano was able to allow the simplification and the translation into the rigorous Weierstrass formulation of limit of the vague but very interesting geometric definition written in the treatise of Maclaurin.

From Peano's formalization of Maclaurin process we can also deduce that $f'(x_0)$ is a number, if it exists, such that the following equivalent definition holds:

$$f'(x_0) = \lim_{x, y \rightarrow x_0} \frac{f(x) - f(y)}{x - y} \quad \text{where } x < x_0 < y.$$

Different Kinds of Derivative and the Mean Theorem

If in the previous relation we do not require that it is $x < x_0 < y$, we obtain a more restrictive definition of derivative than the classical one, since it implies the continuity of the derivative, so for example the function:

$$f(x)=0 \text{ if } x=0 \text{ and } f(x)=x^2\sin\frac{1}{x} \text{ if } x \neq 0 \quad (*)$$

is derivable everywhere in the classical sense, but it is not derivable in $x=0$ in this new sense.⁵ In 1882, October 7, the young assistant Peano, while teaching analysis in the course of his professor Genocchi, writes to him and submits some of his ideas. Among other things he proves that: if $f(x)$ is a function having a finite derivative for every x of an interval (a,b) and if for every α there exists $\varepsilon>0$ such that for every $h\leq\varepsilon$ and for every $x\in(a,b)$ it is $\left|\frac{f(x+h)-f(x)}{h} - f'(x)\right| \leq \alpha$, then $f'(x)$ is a continuous function in (a,b) (Borgato 1991, 68). In (Genocchi 1884, 50) he will observe that this condition is equivalent to the continuity of the derivative.

It is also very interesting to remember that in a letter to the review “Nouvelles Annales de Mathématiques” published in January 1884, Peano corrects the demonstration in the first volume (Jordan 1882-87, 21) of a weak form of the mean theorem, where the derivative is supposed implicitly continuous. In the letter Peano considers the function (*) and observes that, if $a_n = \frac{1}{n\pi}$, then the difference: $\frac{f(a_{2n+1})-f(a_{2n})}{a_{2n+1}-a_{2n}} - f'(a_{2n}) = 0 - (-1) = 1$, similar to differences considered by Jordan and by him thought infinitesimal, is obviously not infinitesimal since $f'(x)$ is not continuous in the origin.

He adds also that it is possible to prove the mean theorem:

if a function $f(x)$ is continuous and derivable in an interval (a,b) , then if x and $x+h$ belong to (a,b) there exists θ such that $0<\theta<1$ and $f(x+h)-f(x) = hf'(x+\theta h)$,

even if $f'(x)$ is not continuous in (a,b) .

Before publishing the letter, the director of the review sent a copy of it to Jordan, who answered saying that he agreed completely with Peano’s criticism; Jordan asked Peano also for a proof of the mean theorem. Then Peano sent privately a letter to Jordan where he proved the mean theorem exactly as it is demonstrated nowadays in the analysis texts for students⁶ (Borgato 1991a, 70-71), (Bottazzini 1994, 170-72).

Only some years after, in 1892, in a brief note on the Belgian review *Mathesis*, Peano will give a new definition of derivative in a point x as the limit of the ratio $\frac{f(x_1)-f(x_2)}{x_1-x_2}$ where x_1 and x_2 both tend to x .

In this way:

a derivative of a function $f(x)$ in an interval is continuous if and only if the incremental ratio $\frac{f(x+h)-f(x)}{h}$ tends to o when h tends to o uniformly with respect to x ; and if and only if the ratio $\frac{f(x_1)-f(x_2)}{x_1-x_2}$ tend to $f'(x)$ when x_1 and x_2 both tend to x , for every x belonging to the interval.

These conclusions are strictly linked to the attempt made in the fifth chapter of (Peano 1887) in order to elaborate a theory of set additive functions: Peano considers coexistent magnitudes, as for example mass and volume of a body, and considers, starting for example from the mean density of a body, its density in its points, considering a process of punctual derivation of a set function. Although this pioneering work was not recognized a long time

⁵ Indeed if we consider $y = \frac{x}{1+\pi x}$ we obtain $x-y = \frac{\pi x^2}{1+\pi x}$ and $\frac{f(x)-f(y)}{x-y} = \frac{[(1+\pi x)^2+1]}{\pi(1+\pi x)} \sin \frac{1}{x}$ whence we deduce that $\lim_{x,y \rightarrow 0} \frac{f(x)-f(y)}{x-y}$ does not exist.

⁶ In a subsequent letter to Gilbert Peano declared he had heard the proof from Genocchi but the demonstration had been given first by Bonnet and had been published after by Serret in his *Cours d’analyse*.

in any way, F. A. Medvedev in 1975 observed that it was more noteworthy than Lebesgue's paper of 1910 that is generally recognized as the source of the modern research on the set additive function theory (Kennedy 1983, 40).

At page 171 of (Peano 1887) the Author proves the following theorem:

The ratio of two coexistent quantities in a point is a continuous function of the point.

And in 1914 he writes:

For the real functions of a real variable, the derivative can exist and be discontinuous. But the derivative in this case is the limit of the incremental ratio, where one extreme is fixed, and the other mobile tending to the first. If by derivative we mean the limit of the incremental ratio where both values given to the variable tend to the same value, as is done for the coexisting quantities, then this derivative, supposed to exist, is a continuous function. (Peano 1914-15, 795)⁷

Peano and Genocchi

Genocchi-Peano treatise was a clear and penetrating look on the analysis, completely in line with the contemporary rigourism period. Not only the *Preface* and the *Annotazioni* but the most part of the book owes very much to Peano,⁸ as he himself admitted in 1889 (Peano 1959):

Using the summaries made by the students to his (Genocchi's) lectures, I compared them point by point with all the main treatises of calculation, and with original Memoirs, thus taking into account the work of many. I made many additions and modifications to his lessons. (Peano 1959, Vol.3, 319)⁹

Anyway, his words did not fail Genocchi's thought. Genocchi,¹⁰ as Peano, was a follower of Weierstrass German school; since 1865, when he started his teaching of

⁷ Per le funzioni reali di variabile reale, la derivata può esistere ed essere discontinua. Ma la derivata in questo caso è il limite del rapporto incrementale, ove un estremo sia fisso, e l'altro mobile tendente al primo. Se per derivata si intendesse il limite del rapporto incrementale ove ambo i valori dati alla variabile tendono a uno stesso valore, come si fa per le grandezze coesistenti, allora questa derivata, supposta esistente, risulta funzione continua.

⁸ Genocchi was in dispute with Peano for this fact. The causes are explained in (Kennedy 1983, 31-34) and (Bottazzini 1994,162-163); in (Carbone, Gatto, Palladino 2001, 211-212) the letter of Genocchi to Cremona dated November 23, 1884 concludes the question, with the affirmation from Genocchi of his high esteem for Peano.

⁹ "Io, servendomi di sunti fatti dagli allievi alle sue (di Genocchi) lezioni, li paragonai punto per punto con tutti i principali trattati di calcolo, e con Memorie originali, tenendo così conto dei lavori di molti. Feci in conseguenza alle sue lezioni molte aggiunte, e qualche modificazione".

¹⁰ The young Peano was taught by Genocchi not only in a strictly mathematical sense, but also he learnt to share Genocchi's interest in the work of famous mathematicians of the past, in particular of Lagrange: an example is given by the following papers by Genocchi: *Di una formula del Leibniz e di una lettera di Lagrange al conte Fagnano*, 1869, Stamperia Reale Torino; *Intorno ad alcune lettere del Lagrange*, 1874, Stamperia Reale Paravia; *Sopra la pubblicazione fatta da Boncompagni di undici lettere di Luigi Lagrange a Leonardo Eulero*, 1877, Roma, *Bullettino di Bibliografia e di Storia delle Scienze mat. e fis.* Tomo X.

Genocchi exposed also the controversy, born in 1846 at Turin Academy and ended in France, after the young professor Chiò had published a paper about the approximation of the roots of a numerical equation that clashed with a theorem by Lagrange: two conceptions of mathematics were faced, the algebraic conception by Lagrange on one side and the conception of the modern analysis, the analysis

differential calculus he had written the notes in his own hand, and continued writing still 1885.¹¹ Inspired to the *Cours d'analyse* by Cauchy, they reveal a deep interest in a rigorous approach to mathematics. Since Genocchi was informed about the existence of continuous functions without derivative only in 1875, they begun one of the subjects of his seminars in 1878, as Peano points up (Peano 1959, 318).

For few years, Peano was interested in analysis, before dedicating all his efforts to his famous *Formulario*, but he gave many other original contributes. For example, in 1890 he published the first example of a fractal, a curve that fills all a plane area; this example was very sensational in the history of the dimension concept. Cantor had proved that it is possible to establish a bijection between an interval and a square but in that period, it was also proved that this bijective application could not be continuous. Now Peano in a short paper on the *Mathematische Annalen* (Peano 1890) proved that it is possible to establish a continuous mapping between an interval and all the points of a square, the mapping being obviously not injective. Peano gave only an arithmetic description of the curve; only many years after, in 1908, he suggested the geometric construction of it and of other analogous curves in the *Formulaire Mathématique*. The first analytical representation of Peano's curve was given by Cesaro in 1897 (Borgato 1991b).

In 1892, among other things, Peano considered examples of functions always increasing and discontinuous in every interval, a topic strictly related to the study of pathological functions as the problem of the derivability of a continuous function (Peano 1892a); he wrote also a paper on the definition of the limit (Peano 1892b), where he proved that the study of the behavior of a function $f(x)$ in a point x_0 , in the case it does not admit a limit, could be simplified considering that in every case, if the function is bounded, a point L exists such that for every σ and for every ε there exists at least one x such that $0 < |x - x_0| < \delta$ and $|f(x) - L| < \varepsilon$. A point as L is nowadays called a *cluster point* and minimum limit and maximum limit in x_0 are respectively the least cluster point and the greatest cluster point.¹²

As Beppo Levi observed in (Levi 1955), Peano's contributions to the theory of functions of real variable is in the first years of his work more didactic than essential, interesting more for the research of simplicity and clarity than for his originality.

However, B. Segre claims that the directives of Bourbakism towards a critical revision and unification of mathematics under the ideas of abstractness and axiomatization were

of Cauchy, supported by Genocchi on the other side: *Di una controversia intorno alla serie del Lagrange*, 1872, Stamperia Reale Torino.

In 1884, date of publication of the denied *Calcolo* edited by Peano, the following note appears: *Due lettere di C. F. Gauss pubblicate dal Principe Boncompagni*, Reale Acc. Sci. Torino. In the memory: *Sur un Mémoire de Daviet de Foncenex et sur les Géométries non euclidiennes*, Reale Acc. Sci. Torino, 1887, Genocchi compared Lagrange's analytic foundation and the functional formulation of mathematics, with great attention also for non-Euclidean geometry. But since the examples of non-Euclidean geometry, pointed out by Professor Beltrami, were given within the Euclidean space, he concluded that the ancient Euclidean geometry is enough for all. In this way a free field was left to analysis.

¹¹ See (Bottazzini 1991) and (Bottazzini 1994), where the notes by Genocchi are exposed.

¹² “[...] the definition of limit now commonly used in the treatises of Analysis for which each function has one limit only, or lacks limit, is a special case of the concept of limit found in Cauchy, Abel and others, according to which each function has some limit values. If these are reduced to one, the particular ordinary definition is given; if there are more, among them there is always the minimum and maximum [...]. Thus generalizing the concept of limit, a vast field is presented open to scholars; it is a question of examining the various propositions known in the particular case of the single limit, and of seeing with what modifications they exist, since they are limits in general” (Peano 1894, 20). For example Peano proves that, given a sequence of real numbers, then the minimum and the maximum limits of the sequence of the arithmetic means lie between the minimum and the maximum limits of the given sequence; the same holds also for the minimum and maximum limit of the sequence of the geometric means (Peano 1894, 21-28).

inspired directly or indirectly by the philosophical convictions of Peano, which were present in all his work and induced him to write his *Formulario Mathematico* (Segre 1955).

Contribution of Peano to Differential Equation Theory

In the years 1886-90, Peano gives new contributions to the study of ordinary differential equations besides the aforesaid discovery of a curve that fills an area. In the note *Sull'integrabilità delle equazioni differenziali del primo ordine* presented in 1886 to the Reale Accademia delle Scienze di Torino, Peano was the first to prove that the Cauchy's problem for the equation $y'=f(x,y)$ has solution also if the function $f(x,y)$ is only continuous, by means of a new conception of the problem. Perron some years after went back to Peano's proof again and for a long time Peano took the credit for it. In 1890, Peano faced the same problem from a quite different point of view, first of all considering a system of first order equations. Three years after, a free re-exposition of Peano's Memory was given, removing the obstacle to the reading posed by the use of the logic ideography and the *excess of precision* Peano had introduced in his exposition; then the demonstration procedure was universally esteemed and motivated new studies by de la Vallée Poussin, Arzelà and Osgood.

Beppo Levi remembers also the Note: *Integrazione per serie delle equazioni differenziali lineari*, on the Reale Accademia delle Scienze di Torino, (published in 1888 on *Math. Ann.*) where Peano considered a system of n first order differential equations in n unknown functions:

$$(*) \quad \frac{dx_i}{dt} = \sum_{j=1}^n r_{ij} x_j;$$

In this paper he exposed an integration procedure alike to that presented by Picard four years after; Picard considered a more general not linear system $\frac{dx_i}{dt} = f_i(x_1, \dots, x_n, t)$ and applied a method he called *subsequent integrations method*. Peano claimed some years after, in 1897, his partial priority. As Beppo Levi observes it is difficult to compare the origins of the methods used by the two authors. It is more significant to remember how Peano relates his method to the linear operator calculus: indeed if $X=(x_1, \dots, x_n)$, by (*) we can say that $\frac{dX}{dt}$ is obtained from X by means of a linear substitution R , function of the real variable t , that is:

$$\frac{dX}{dt} = RX.$$

The integration by series can be seen as the analogous integration of the simple equation, for $n=1$, $\frac{dx}{dt} = rx$ obtained by posing $x(t)=x(t_0)e^{\int_{t_0}^t r(s)ds}$, whence we can write $x(t)$ as a sum of a series (See Cassina 1933).

Peano was a friend of Ernesto Cesaro (1859-1906), favourite disciple of the Belgian professor Eugène Catalan and professor of Analysis at the University of Naples. Their friendship begun with a letter of January 20, 1892 with some observations by Cesaro about the Genocchi-Peano treatise. The correspondence continued with an exchange of ideas and manifestations of mutual esteem; it is also interesting for the notes and the books Cesaro sent with regularity. For example, in the letter October 31, 1894 Peano thanks Cesaro for his note *Sulla geometria intrinseca delle congruenze* and for the book *Introduzione alla teoria matematica dell'elasticità*. In the letter of November 5, 1896 Peano thanks Cesaro for the book *Lezioni di Geometria Intrinseca* he finds very interesting and writes he is very busy since the second volume of his *Formulaire de Mathématiques* is in press; only two years after, letter of November 22, 1898, Peano writes he has received Cesaro's treatise *Elementi di Calcolo Infinitesimale*, a noteworthy book, where the last researches of Cesaro are exposed with rigour and elegance. And one year after, letter of December 10, 1899, Peano writes: "Your

works lend themselves very well to translation into symbols, due to their precision and clarity” (“I suoi lavori si prestano assai bene alla traduzione in simboli, a causa della loro precisione e chiarezza”); and proposes to Cesaro to translate some propositions of his treatises in the symbology of the *Formulario* (Palladino 2000). E. Cesaro, even if it is not enough mentioned today (only by his famous theorems about series, the media theorem known as Cesaro Stolz theorem) is another example of the interest shared by some Italian mathematician in the new conceptions about functions and mathematical objects, and was very much esteemed by others contemporaneous mathematicians as Volterra, Segre, Beltrami, Hermite [...] (Palladino 2000).

In the same year of the second edition of his *Elementi di calcolo infinitesimale*, in 1905, Cesaro published also a paper where he gave a new construction of the Koch curve, one of the first examples of fractals, another testimony of the similarity of his interests and even perhaps of character with his friend Peano.

Term by Term Integration and Reduction of Double Integrals

In 1970 Heine had introduced uniform convergence as a sufficient condition in order the limit of a convergent sequence of continuous functions was continuous too. The Italian mathematician C. Arzelà (1847-1912), another student at the *Scuola Normale* of Pisa in the early seventies, thought at first that the uniform convergence was not only a sufficient but also a necessary condition, however by means of counterexamples he convinced himself that this was not the case and he faced the problem of the determination of the weakest type of convergence of a sequence of real continuous functions in order that the limit was continuous too.

In 1883-84 he characterized this type of convergence. In 1885, in a series of four papers published on the *Rendiconti della Accademia dei Lincei* he introduces a fundamental Lemma, he uses to establish the required necessary and sufficient condition (Arzelà 1885a).

Lemma - Suppose that for every $n \in \mathbb{N}$ there exist intervals in (a, b) in finite number whose union U_n has length equal to b_n ; then if for every n it is $b_n \geq d$, where $d > 0$ is a fixed number, there exists $x_0 \in (a, b)$ that belongs to infinitely many U_n .

Accordingly, given the sequence f_n converging to f in (a, b) and $\sigma > 0$, Arzelà considered the set:

$$E(n, \sigma) = \{x : |f_n(x) - f(x)| > \sigma\}$$

and proved that if $E(n, \sigma)$ is a finite union of intervals then for every $\sigma > 0$ it is

$$(*) \quad \lim_{n \rightarrow \infty} C_e(E(n, \sigma)) = 0,$$

where C_e stands for the total length (Arzelà 1885a, 267). Indeed if there exists $\sigma > 0$ such that $(*)$ does not hold there exist $d > 0$ and infinitely many n such that $C_e(E(n, \sigma)) > d$ and therefore by the Lemma there exists x_0 that belongs to infinitely many $E(n, \sigma)$, that is in contrast with $f(x_0) = \lim f_n(x_0)$.

So Arzelà proves something similar to the theorem that if a sequence of measurable functions almost everywhere converges in a bounded interval (a, b) then it converges in measure in (a, b) , even if Arzelà did not use a general concept of measure yet. Perhaps he was inspired by the fact that the requirement of something similar to convergence in measure, with respect to outer content, had appeared briefly for the first time in 1878 in a paper about the problem of term-by-term integration by Kronecker (Hawkins 2002, 111-2). In any way we will encounter only in 1903, after the fundamental concept of measure will be given by Lebesgue, the modern definition of convergence in measure in (Lebesgue 1903),

and the implication a.e. convergence implies convergence in measure will be proved in the book by Lebesgue on trigonometric series (Lebesgue 1906, 6) and also in (Borel 1905, 27).

Then Arzelà proves the following theorem about the continuity of the limit of a sequence of continuous functions (Arzelà 1899-1900, 142):

Theorem 1. Let $f_n(x)$ be a converging sequence of functions defined in the interval (a,b) , all continuous in $x_0 \in (a,b)$. Then the limit $f(x)$ is continuous in x_0 if and only if for every $\varepsilon > 0$ there exists $\nu \in \mathbb{N}$ such that for every $n \geq \nu$ there exists $\delta > 0$, depending on n , such that $|f_n(x) - f(x)| < \varepsilon$ for every $x \in]x_0 - \delta, x_0 + \delta[$.

Arzelà proves also (Arzelà 1899-1900, 150-513):

Theorem 2. Let $f_n: (a,b) \rightarrow \mathbb{R}$ be a converging sequence of continuous functions and let $f(x)$ its limit. Then $f(x)$ is continuous if and only if for every $\sigma > 0$ and for every $\nu \in \mathbb{N}$, there exists a finite set of indices $n_1, \dots, n_s > \nu$ and corresponding sets $U_{n_i}, i \in \{1, \dots, s\}$, everyone being a finite union of intervals, such that $\cup U_{n_i} = (a,b)$ and for every $x \in (a,b)$ there is $i \in \{1, \dots, s\}$ such that $x \in U_{n_i}$ and $|f_{n_i}(x) - f(x)| < \sigma$.

In this theorem the sequence converges in a particular way Arzelà defines step-by-step uniform convergence. Arzelà considers some examples:

1 – The sum of the series $(1-x) + (1-x)x + \dots + (1-x)x^n + \dots, 0 \leq x \leq 1$, is the discontinuous function $S(x) = 1$ for $0 \leq x < 1, S(1) = 0$. It is $S_n(x) = 1 - x^{n+1}$, a sequence not uniformly convergent in $[0,1]$; also Arzelà's theorem does not apply.

Indeed let $\sigma \in (0,1)$ and $\delta \in (0,1)$. Let $\sigma' = 1 - \sigma$: then in the interval $(1-\delta, 1]$ there exists x such that for enough large n it is $x < \sqrt[n]{\sigma'}$ since $\lim_{n \rightarrow \infty} \sqrt[n]{\sigma'} = 1$. Then $x^n < \sigma'$ and $S_n(x) - S(x) = 1 - x^{n+1} > 1 - \sigma' = \sigma$, in spite of Theorem 1.

2. Let, for $x \in (0,1)$: $u_n(x) = \frac{nx}{1+n^2x^2} \frac{(n+1)x}{1+(n+1)^2x^2}$. Then the sum is $S(x) = \sum_{n=0}^{\infty} u_n(x) = 0$, a continuous function, even if the convergence is not uniform. And in this case Arzelà's Theorem 1 applies:

indeed for every $n \in \mathbb{N}$ it is $R_n(x) = \frac{nx}{1+n^2x^2} > \sigma$ for every x belonging to $(\frac{1-\sqrt{1-4\sigma^2}}{2n\sigma}, \frac{1+\sqrt{1-4\sigma^2}}{2n\sigma})$ ($\sigma < 1/2$).

Therefore for every $1/2 > \sigma > 0$ and for every $n \in \mathbb{N}$ there exists an interval whose length is $\frac{\sqrt{1-4\sigma^2}}{n\sigma}$ in which $R_n(x) > \sigma$. Since $\lim_{n \rightarrow \infty} \frac{\sqrt{1-4\sigma^2}}{n\sigma} = 0$, we conclude that for every $x_0 > 0$ and $\sigma < 1/2$ it is possible to determine $\nu \in \mathbb{N}$ such that for every $n > \nu$ there exists $\delta > 0$ such that $R_n(x) < \sigma$ for every $x \in]x_0 - \delta, x_0 + \delta[$; if $x_0 = 0$, for every n there exists $\delta > 0$ such that $[0, \delta[$ is disjoint from $(\frac{1-\sqrt{1-4\sigma^2}}{2n\sigma}, \frac{1+\sqrt{1-4\sigma^2}}{2n\sigma})$ and therefore for every $x \in [0, \delta[$ it is $R_n(x) < \sigma$; so, in accordance with Arzelà's Theorem 1, $S(x)$ is continuous for every x_0 .

Arzelà faced also the problem of the determination of the necessary and sufficient condition in order that the limit of a sequence of integrable functions is integrable (Arzelà 1885b). If the sequence is uniformly convergent then the limit is integrable too, but if the convergence is not uniform it may be so or not.

Example 1 - Baire in his thesis, some years after, in order to give an example of a function of class 2, considered the following example of uniformly bounded functions (Baire 1899, 69). Let $f_n: (0,1) \rightarrow (0,1)$ be such that $f_n(x)=0$ if $x=p/q$, with coprime p and q , $q \leq n$; $f_n(x)=1$ otherwise. Then for every n it is $f_n(x)=1$ for every x except finitely many points and therefore $f_n(x)$ is integrable. The limit function, a Dirichlet function, assumes the value 0 if x is rational, 1 if x is irrational, thus it is totally discontinuous and therefore not integrable.

Observe that in the previous example (*) is not true for every σ , accordingly with the fact that, as Arzelà proves, if a convergent sequence of integrable functions is uniformly bounded then if the limit is integrable (*) holds. Arzelà thought that in general (*) does not hold, and therefore the limit is not integrable, as in Example 1, but he did not give any example.

Condition (*) allows to obtain a condition exposed in (Arzelà 1885 b, 325-326) we can summarize in the following way:

Theorem 3. Let $f_n: (a,b) \rightarrow \mathbb{R}$ be a convergent sequence of integrable functions and let $f(x)$ its limit. Then $f(x)$ is integrable if and only if for every $\sigma > 0$, for every $\varepsilon > 0$ and for every $n \in \mathbb{N}$ it is possible to remove from (a,b) a finite number of intervals whose union D has length $< \varepsilon$ and there is a finite set of indices $n_1, \dots, n_s > n$ and corresponding sets U_{n_i} , $i \in \{1, \dots, s\}$, everyone being a finite union of intervals, such that $\cup U_{n_i} = (a,b) - D$ and for every $x \in (a,b) - D$ there is $i \in \{1, \dots, s\}$ such that $x \in U_{n_i}$ and $|f_{n_i}(x) - f(x)| < \sigma$.

Arzelà calls *general step-by-step uniform convergence* this type of convergence (Arzelà 1885b, 326). Clearly it is weaker than uniform convergence but stronger than the convergence that after Lebesgue will be said a.e. uniform convergence.

Note 1 - Let $f_n: (a,b) \rightarrow \mathbb{R}$ be a convergent sequence of integrable functions and let $f(x)$ its limit. Then if $f(x)$ is integrable the sets $E(n, \sigma)$ are all Peano-Jordan measurable by (Biacino 2015) and therefore it is not restrictive, in order to prove (*), to suppose they are unions of a finite number of intervals.

Indeed for every $n \in \mathbb{N}$ and every $\sigma > 0$ there exists a finite union of intervals, $P_{n,\sigma} \subseteq E(n, \sigma)$ such that $\mu(E(n, \sigma)) < \mu(P_{n,\sigma}) + 1/n$, (μ Peano-Jordan measure), whence (*). Indeed if $\lim_n \mu(E(n, \sigma)) > 0$ then there exist $d > 0$ and infinitely many $n \in \mathbb{N}$ such that $\mu(E(n, \sigma)) > d$, therefore $\mu(P_{n,\sigma}) + 1/n > d$ and for enough large n , $\mu(P_{n,\sigma}) > d/2$; then the proof follows as above.

This simplifies the very complex proof by Arzelà that if f is integrable then $E(n, \sigma)$ is a finite union of intervals and therefore (*) holds (Arzelà 1885b, 321-323).

Observe also that “integrability” stands in this case in place of “measurability” in the classical implication of Lebesgue measure theory.

Note 2 - If f is integrable then (*) holds and by this fact Arzelà is allowed to prove that the condition in Theorem 3 is necessary. But (*) does not make sure that the same condition is also sufficient if the sequence is not uniformly bounded. There are convergent sequences of integrable functions for which (*) holds but the limit is not integrable, as in the following example (not given by Arzelà):

Example 2 - Let $f_n(x) = 1/(x+1/n)$ if $x \in (0,1)$. These functions are all integrable, their limit is the function $f(x) = 1/x$ for $x \in (0,1)$, that is not integrable, but $\lim_n C_\varepsilon(E(n, \sigma)) = 0$.

Now there are examples proving that even if the limit of a sequence of integrable functions is integrable, it can be not true that it is possible to integrate term by term (see the following Example 3).

In (Arzelà 1899-1900) two different cases are considered. In the first case the functions are supposed uniformly bounded and the following proposition is established:

Theorem 4. *If the functions f_n and f are uniformly bounded and are integrable then if (*) holds it is possible to pass to the limit under the integral sign.*

Indeed if (*) holds then for every (x_0, x_1) contained in (a, b) :

$$\left| \int_{x_0}^{x_1} (f_n - f) \right| \leq \int_{x_0}^{x_1} |f_n - f| \leq \int_{E(n, \sigma)} |f_n - f| + \int_a^b \sigma \leq 2M C_e(E(n, \sigma)) + \sigma(b-a),$$

where $M > 0$ is such that $|f_n(x)| \leq M$ for every $n \in \mathbb{N}$ and for every $x \in (a, b)$.

Given $\varepsilon > 0$, let $\sigma < \frac{\varepsilon}{2(b-a)}$ and let ν be such that for $n > \nu$ it is $C_e(E(n, \sigma)) < \frac{\varepsilon}{4M}$. Then for per $n > \nu$ it is

$$\left| \int_{x_0}^{x_1} (f_n - f) \right| < \varepsilon, \quad \text{that is} \quad \lim_n \int_{x_0}^{x_1} f_n = \int_{x_0}^{x_1} f \quad (\text{Arzelà 1889-1900, 722-24}).$$

If the functions are not uniformly bounded the preceding argument does not apply. First Arzelà thought that if the sequence of integrable functions $f_n(x)$ is not uniformly bounded in the interval (a, b) but its limit $f(x)$ is integrable then it was possible to pass the limit under the integral if and only if $\lim_n \int_a^x f_n(t) dt$ was a continuous function (Arzelà 1885c, 532-33). But the American mathematician William Fogg Osgood (1864-1943) was in that period working about the same argument and two years after he published an important paper (Osgood 1897, 167-170) where among other things he gave a counter example about the previous question: in fact in that paper a series is considered such that the series of the integrals is everywhere zero and therefore continuous but it does not coincide anywhere with the integral of the sum except in one point. Thus in (Arzelà 1899-1900) the question is considered again; Osgood's example is quoted and at page 169 the sequence of the elements of the series considered by Osgood is proved to converge uniformly step by step. The theorem partially erroneous is corrected and the theorem by (Osgood 1897, 188) about a sufficient condition for the integrability of a series term by term, slightly improved (Osgood considers only continuous functions), is fundamentally reported as follows (Arzelà 1899-1900, 733).

Theorem 5. *Let $u_n(x)$ be, for every natural n , an integrable function in the interval (a, b) and let $f_s(x) = \sum_{n=1}^s u_n(x)$, $f(x) = \sum_{n=1}^{\infty} u_n(x)$. Then it is a sufficient condition for the integrability of the series term by term that: the series $\sum_{n=1}^{\infty} \int_a^x u_n(t) dt$ is an integrable function of x and the set G of the points x' such that it is impossible to determine a real number $L > 0$, a neighbourhood $(x' - \delta, x' + \delta)$ and a natural number n such that $|\sum_{s=n}^{n+m} u_s(t)| < L$ for every $t \in (x' - \delta, x' + \delta)$ and for every natural m , is enumerable.*

In the example by Osgood the points x' as in Theorem 5 constitute a set of the power of the continuum. But Arzelà at page 731 observes that the condition that G is enumerable is not sufficient.

We conclude this exposition exhibiting the following example in which a step by step uniformly convergent sequence is studied.

Example 3 – In (Arzelà 1899-1900, & 4) the not uniformly bounded functions $f_n(x) = -2n^2 x \exp(-x^2 n^2)$ are considered, re-examining an example already given in (Darboux 1875). The sequence is convergent to the integrable function $f(x) = 0$ for every x , it is not uniformly convergent, but

it is step by step uniformly convergent. Arzelà studies the functions with a fixed n in an interval $(-b, b)$ (the figure is clipped from (Arzelà 1899-1900, 163): here it has been partially corrected). Consider the interval $E_n = (\frac{1}{n^3}, \frac{1}{n^3})$ where the function f_n is decreasing. Since $|f_n(\frac{1}{n^3})| < \frac{2}{n}$, we have that

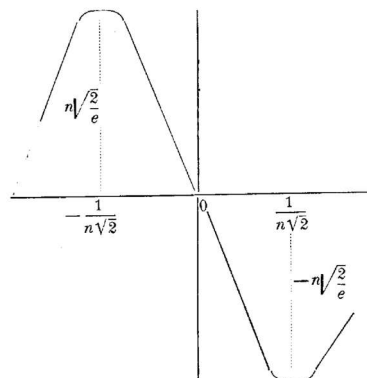
$$(**) \quad |f_n(x)| < \sigma \quad \text{for every } x \in (\frac{1}{n^3}, \frac{1}{n^3}) \text{ if } n > \frac{2}{\sigma}.$$

It is not difficult to prove that inequality $(**)$ holds also for $x \geq \frac{1}{m} + \frac{1}{\sqrt{m}}$ and for $x \leq -\frac{1}{m} - \frac{1}{\sqrt{m}}$ if $n=m$ is sufficiently large. Then for $n=m$ sufficiently large we have two intervals

$$(-b, -\frac{1}{m} - \frac{1}{\sqrt{m}}), (\frac{1}{m} + \frac{1}{\sqrt{m}}, b),$$

such that in every point of their union, D_m , $(**)$ holds. It is easy to see that it is possible to determine p such that $E_m \cup D_{m+p} = (-b, b)$ and therefore for every $x \in (-b, b)$ it is $x \in E_m$ or $x \in D_{m+p}$: in the first case $|f_m(x)| < \sigma$; in the second case $|f_{m+p}(x)| < \sigma$, that is the sequence is step by step uniformly convergent to 0: this fact is obviously implied also by the continuity of the limit, thanks to Theorem 2.

Observe that if $F(t) = \lim_n \int_0^t -2n^2 x e^{-n^2 x^2} dx$, then $F(0) = 0$ and $F(t) = -1$ for $t \neq 0$, that is $F(t)$ is not continuous in accordance with the fact that it is not possible to pass to the limit under the integral sign; even though $(*)$ holds.



A great merit of Lebesgue integration theory was to give a simple general sufficient condition in order to integrate term by term; and Vitali characterized completely, in the framework of Lebesgue integration theory, the series for which it is possible to pass to the limit under the integral sign.

C. Arzelà was interested in the integration of the functions of two variables in 1891 (Arzelà 1891). He requires that the domain E of integration is bounded by a continuous closed simple and rectifiable curve; the last hypothesis is added since in the preceding year Peano has discovered continuous curves that pass through every point of a square of the plane. Peano himself has claimed that some other hypothesis had to be added on the boundary in order it can be enclosed in a region of arbitrarily small area. Now in order that a double integral can be calculated by two iterated simple integrations it is necessary that the intersections E_c of E with the straight lines of equation $y=C$ are suitable for integration. This is why at first Harnack postulated that they were intervals and subsequently, in a paper on *Mathematische Annalen*, in 1886, thought that it was enough that E_c were equal to the union of a finite number of intervals. For this reason, it may be Arzelà requires that the boundary is

not only a continuous closed and simple curve, but also a rectifiable one. Now it can be that the boundary is a continuous rectifiable simple closed curve but the E_c are not the union of a finite number of intervals.

Indeed consider the Volterra's function $y=f(x)$ and the set $E=\{(x,y): 0 \leq x \leq 1; -1 \leq y \leq f(x)\}$; since $f'(x)$ is bounded, $f(x)$ is of bounded variation and the hypotheses of Arzelà are verified. If we consider $y=0$ the set E_0 coincides with the union of the nowhere dense subset R , considered before, and an infinite set of points in the interior of the complement of R : this set therefore is not the union of a finite number of intervals as Arzelà thought. The question would be solved by introducing the concept of measurable set: as we have already seen Peano claimed in 1887 that in this case the boundary has null content. Jordan in 1893 would close the question.

In 1891 Arzelà is concerned with the problem of the reduction of a double integral into two consecutive simple integrations: In that period this problem was the object of some controversies between Harnack and Stolz, on the *Mathematische Annalen*. Arzelà remembers Thomae example on *Zeitschrift für Mathematik und Physik*, of the function:

$$f(x,y)=1 \text{ if } x \text{ is a rational number, } f(x,y)=2y \text{ if } x \text{ is irrational.}$$

It is clear that $\int_0^1 dx \int_0^1 f(x,y)dy=1$ but $\int_0^1 dy \int_0^1 f(x,y)dx$ does not exist.

In order to solve the question Arzelà gives a particular definition of a function that is *uniformly integrable with respect to x* and *uniformly integrable with respect to y* and proves that under these hypotheses the two iterated integrals do exist and are equal.

Conclusion

As we have seen in the preceding pages, in the last decades of nineteenth century the new generation of Italian mathematicians, who had mainly formed at Pisa, with the fundamental guidance of Dini and Betti, at Pavia with the guidance of Brioschi and Casorati and at Turin under the guidance of Genocchi, begun to produce important results at an international level in real analysis, measuring themselves mainly with the disciples of Dirichlet, Riemann, Weierstrass. Besides the results considered in this paper they gave very many other important contributions. Arzelà after giving a systematic exposition of his results about the minimal condition assuring the continuity of the sum of a series of continuous real function or the possibility to integrate term by term a series of integrable functions, as we have exposed, continued his research in his memory: *Sulle serie di funzioni analitiche*, published in 1903 on the *Rendiconti Accademia delle Scienze di Bologna*: also in this case the problem, on which also Vitali, Osgood and Montel worked, was to determine the minimal conditions under which a series of analytic function in some field of the complex plane converges to a holomorphic function. Using Arzelà's techniques Vitali obtained a famous theorem of compactness that was successively generalized by Montel.

Also for another question great achievements were gained by Italian mathematics at the end of nineteenth century. Volterra, after the interesting papers he wrote in 1881 with the guidance of Dini, when he was still a student at Pisa, and after the degree in mathematics, did not follow Dini's direction any more, but fascinated by the courses of mathematical physic and mechanics of E. Betti, he became extraordinary professor of mechanics in the same university in 1883, and begun to make important pioneering researches in the framework of applied mathematics, integral and differential equations. He observed that in many problems of physic and mechanics one has to consider some quantities that depend on all the values one or more functions assume in a given field. For example the temperature in a point of a conducting thin layer of metal depends on all the values the temperature has at the boundary; also the area of a region bounded by a curve depends on the particular curve, the length of a rectifiable curve depends on the curve and so on. If with every curve lying in the

given field a real number is associated then a new type of function arises, for which Volterra, used the words “function of line”. He exposed his researches about this topic in (Volterra 1887) that is considered unanimously as the birth certificate of a new field of mathematical research: the functional analysis. In it Volterra extends all the notions of infinitesimal calculus, defines the concept of continuity for a function of line, he studies its variations and also defines the derivatives of whatever order, the Taylor series, extends the theory of implicit functions and so on. The following step was made by Frechet, inspired by Volterra, and consisted in considering as a variable no more a line, but whatever element to be defined as a point. The field of these points was called space, or also, for its nature, abstract space.

Also the paper of 1883 by Giulio Ascoli would be fundamental some years after for functional analysis (Ascoli 1883): it is the study about the limiting curve of a given family of curves, were, given an aggregate F of functions all defined in a closed and bounded interval $[a,b]$, he called them *uniformly equi-continuous* if for every $\varepsilon > 0$ it is possible to find $\delta > 0$ such that $|f(x+h)-f(x)| < \varepsilon$ for every $|h| < \delta$, for every $x \in [a,b]$ and for every $f \in F$. The functions are called also *uniformly bounded* if there exists $M > 0$ such that $|f(x)| \leq M$ for every $x \in [a,b]$ and for every $f \in F$ (see also (Cinquini 1970)). The following theorem was found by Ascoli (Ascoli 1883, 547-549).

If F is an aggregate of uniformly equi-continuous and uniformly bounded functions defined in the interval $[a,b]$, then it is possible to extract from F a uniformly convergent sequence.

The subsequent treatment by Ascoli is very complex and general. He considers, in the hypotheses that the functions of the family F are equi-continuous and uniformly bounded, the family of the limit points, he calls *prime derivative* of the given family, which constitute in general a family of equi-continuous and uniformly bounded functions and therefore possesses in general limit points and so on until the p^{th} derivative. If the p^{th} derivative consists in a finite number of curves, the family F is of order p . If for any p this does not happen the family has no order. If a family has order p then it can be enclosed in an arbitrarily small space (Ascoli 1883, 551). Subtle considerations of topological character mingle continually.

Then Ascoli considers some particular family of functions: for example he considers the family of the functions derivable in an interval (a,b) and such that the derivatives are positive and increasing and he shows that such derivatives are uniformly bounded in every closed interval contained in (a,b) . He proves that consequently the previous derivatives are also equi-continuous. Let V be the set of the derivatives we are considering, if r is its order, let K be a limit curve of the aggregate of the $(r-1)^{\text{th}}$ derivative of V : let $y=h(x)$ be the equation of K . Ascoli proves that $h(x)$ can be always increasing, always decreasing, first constant and after increasing or decreasing and also before decreasing or increasing and after constant and not for example first constant, then increasing and after constant again (Ascoli 1883, 558). Similar properties are determined for other families of derivatives. The proof depends on the following theorem:

Let $f_n(x)$ be a sequence of functions all continuous in an open interval (a,b) ; let $f_n(x)$ be always increasing, decreasing or constant, with constant sign and let $\lim_n \int_a^x f_n(t) dt = 0$ for every $x \in]a,b[$ (and therefore all the functions are Cauchy integrable); then $\lim_n f_n(x) = 0$ for every $x \in]a,b[$.

Ascoli proves, among other things, in Nota III, that the diagram of an increasing and continuous curve is rectifiable; it is an original result, since the studies about the rectification of curves begin with Ludwig Scheeffer in 1884 with a paper on t. 5 of Acta Mathematica and continue with Jordan in 1893 on the second edition of his Cours d'Analyse (but these authors study general plane curve also not continuous).

In (Arzelà 1899-1900, 171 and following) we can read in a more concise manner the definition and demonstration of Ascoli- Arzelà theorem. Moreover in (Arzelà 1899-1900, 182) a sufficient condition is also given.

The functions of a family F all defined in an interval (a,b) are uniformly equi-continuous if there exist two numbers l and L such that:

$$l < \frac{f(x_1)-f(x_2)}{x_1-x_2} < L \quad \text{for every } f \in F \text{ and for every } x_1 \text{ and } x_2 \in (a,b).$$

Arzelà would use all the previous considerations in the framework of the functions of line: in his studies about the so called Dirichlet's principle (if a function increases and then decreases or viceversa then it has a maximum or a minimum) he proved that given a functional G defined in an aggregate F of uniformly equi-continuous and uniformly bounded functions, if G is continuous then, as in the Weierstrass theorem for the ordinary functions, there exist functions $h \in F$ and $k \in F$ such that $G(h) \geq G(f)$ and $G(k) \leq G(f)$ for every $f \in F$. This would be considered as the starting point for the direct method of variational calculus which would have his modern develop in twentieth century.

Then we have seen that at the end of nineteenth century many professors of mathematics of the most important Italian Universities: Pisa, Turin, Padua, Pavia, Bologna, Naples, Milan, were men open to the new trends of research and actively integrated in the international community of mathematicians. The successive generation, the generation of Vitali, Fubini, Beppo Levi, Tonelli, Severini, ... , would give in turn very important results in real analysis and its applications, which would grant them a respected place in the history of mathematics.

References

- Arzelà, Cesare. 1885 a. Un teorema intorno a una serie di funzioni. *Rend. Acc. dei Lincei*, 4 (1): 262-67.
- Arzelà, Cesare. 1885 b. Sulla integrabilità di una serie di funzioni. *Rend. Acc. dei Lincei*, 4 (1): 321-26.
- Arzelà, Cesare. 1885 c. Sulla integrazione per serie. *Rend. Acc. dei Lincei*, 4 (1): 532-37, 566-69.
- Arzelà, Cesare. 1891, Sugli integrali doppi. *Mem. Acc. Sci. Bologna*, 5 (2): 133-47.
- Arzelà, Cesare. 1899-1900. Sulle serie di funzioni. *Mem. Acc. Sci. Bologna*, 5 (8): 131-86, 701-44.
- Ascoli, Giulio. 1875. Sul concetto di integrale definito. *Rend. R. Acc. Lincei*, 2 (2): 863-872.
- Ascoli, Giulio. 1883. Le curve limiti di una varietà data di curve. *Memorie R. Accademia dei Lincei*, Vol. XVIII, 521-586.
- Baire, Renè. 1899, Sur les fonctions de variables réelles. *Annali di Matematica*, 3 (3): 1-123.
- Bettazzi, Rodolfo. 1884. Sui concetti di derivazione e di integrazione delle funzioni di più variabili. *Giorn. Mat.*, (22): 133-67.
- Biacino, Loredana. 2015. Una caratterizzazione delle funzioni integrabili secondo Riemann. *Per. Math.*, (1): 75-79.
- Biacino, Loredana. 2018. The Concept of Function at the Beginning of the 20th Century: A Historiographical Approach. *Transversal: International Journal for the Historiography of Science*, (5): 171-192.
- Borel, Emile. 1905. *Leçons sur les fonctions de variables réelles et leur développements par des séries de polynomes*. Paris, Gauthier – Villars.
- Borgato, Maria Teresa. 1991a. Alcune lettere inedite di Peano a Genocchi e a Jordan sui fondamenti dell'analisi. *Angelo Genocchi e i suoi interlocutori scientifici, a cura di Alberto Conte e Livia Giacardi, Università di Torino*, 61-97.
- Borgato, Maria Teresa. 1991b. Giuseppe Peano: tra analisi e geometria. *Atti Convegno Peano e i fondamenti della matematica. Modena 22-24 ottobre 1991*, 139-169.

- Borgato, Maria Teresa. 2009. Continuity and discontinuity in Italian mathematics after the unification: from Brioschi to Peano. *Organon* (41): 219-231.
- Bottazzini, Umberto. 1991. Angelo Genocchi e i principi del calcolo. *Angelo Genocchi e i suoi interlocutori scientifici, a cura di Alberto Conte e Livia Giacardi, Università di Torino*, 31-60.
- Bottazzini, Umberto. 1994. *Và pensiero. Immagini della matematica nell'Italia dell'Ottocento*. Il Mulino.
- Carbone, Luciano; Gatto, Romano; Palladino Franco (a cura di). 2001. L'epistolario Cremona-Genocchi (1860-1866). Olschki.
- Cassina, Ugo. 1933. L'opera scientifica di Giuseppe Peano. *Rend. Sem. Mat. Fis. Milano*, (7): 323-389.
- Cassina, Ugo. 1950. L'area di una superficie curva nel carteggio inedito di Genocchi con Schwarz ed Hermite. *Rend. Ist. Lombardo* (83): 311-328.
- Cinquini, Silvio. 1970. Giulio Ascoli, Cesare Arzelà e le funzioni egualmente continue. *Rend. Parte generale Ist. Lomb. Acc. Sci. Lett* (104): 3-12.
- Darboux, Gaston. 1875. Mémoire sur les fonctions discontinues, *Ann. École Norm.*, 2 série, IV, 55-112.
- Dini, Ulisse. 1878 [1990]. *Fondamenti per la teorica delle funzioni di variabili reali*. Nistri, Pisa, Edizioni U.M.I.
- Genocchi, Angelo. 1884. *Calcolo differenziale e principi di calcolo integrale pubblicati con aggiunte del Dr. Giuseppe Peano*. Roma-Torino, Fratelli Bocca.
- Hawkins, Thomas. 2002. *Lebesgue's theory of integration. Its origins and developments*. AMS Chelsea Publishing.
- Jordan, Camille. 1882-87. *Cours d'analyse de l'Ecole Polytechnique*. 1st ed. 3 vols. Paris: Gauthier-Villars.
- Jordan, Camille. 1893-96. *Cours d'analyse de l'Ecole Polytechnique*. 2nd ed 3 vols. Paris: Gauthier-Villars.
- Kennedy, Hubert .1983. *Peano. Storia di un matematico*. Boringhieri.
- Lebesgue, Henry. 1903. Sur une propriété des fonctions. *C. R. Acad. Sci. Paris*, Vol. 137, 1228-1230.
- Lebesgue, Henry. 1904a [1928]. *Leçons sur l'intégration et la recherche des fonctions primitives*. Paris, Gauthier-Villars, Deuxième édition.
- Lebesgue, Henry. 1904 b. Sur une propriété caractéristique des fonctions de classe 1. *Bull. Soc. Math*, 229-242.
- Lebesgue, Henry. 1906. *Leçons sur les séries trigonométriques*. Paris : Gauthier – Villars.
- Lebesgue, Henry. 1910. Sur l'intégration des fonctions discontinues. *Ann. E.N.S.* (3), t.27, 361-450.
- Letta, Giorgio. 1994. Le condizioni di Riemann per l'integrabilità e il loro influsso sulla nascita del concetto di misura. *Rend. Acc. Naz. Sci., detta dei XL. Serie V, Vol. XVIII, Parte I*, 143-69.
- Levi, Beppo. 1955. L'opera matematica di Giuseppe Peano. *In memoria di Giuseppe Peano, Celebrazione di Giuseppe Peano, Liceo Scientifico di Cuneo*, 9-21.
- Maclaurin, Colin. 1742. *A treatise of fluxions*. Vol.II, Edinburgh: Ruddimans.
- Osgood, William F. Non-Uniform Convergence and the Integration of Series Term by Term. *Am. Journ. of Math.*, 19 (2): 155-190.
- Palladino, Franco. 2000. *Le corrispondenze epistolari tra Peano e Cesaro e Peano e Amodeo*. Quaderni P. RI. ST. EM.
- Peano, Giuseppe. 1883, Sulla integrabilità delle funzioni. *Atti Acc. Sci. Torino*. (18): 439-46.
- Peano, Giuseppe. 1887. *Applicazioni geometriche del calcolo infinitesimale*. Torino: Bocca.
- Peano, Giuseppe. 1890. Sur une courbe qui remplit une aire. *Math. Ann.* (36): 157-60.
- Peano, Giuseppe. 1892 a. Esempi di funzioni sempre crescenti e discontinue in ogni intervallo. *Rivista di Matematica*. (2): 41-2.

- Peano, Giuseppe. 1892 b. Sulla definizione del limite di una funzione. *Rivista di Matematica*. (2): 77-9.
- Peano, Giuseppe. 1894. Estensione di alcuni teoremi di Cauchy sui limiti. *Atti R. Acc. Sci. Torino*, (XXX): 20-41.
- Peano, Giuseppe. 1914-15. Le grandezze coesistenti di Cauchy. *Atti Acc. Scienze Torino*. (50): 786-795.
- Peano, Giuseppe. 1959. *Opere Scelte*. Vol. III, Cremonese. 317-20.
- Segre, Beniamino. 1955. Peano e il Bourbakismo. In memoria di Giuseppe Peano, *Celebrazione di Giuseppe Peano, Liceo Scientifico di Cuneo*, 31-39.
- Severini, Carlo. 1897. Sulla rappresentazione analitica di alcune funzioni reali discontinue di variabili reali. *Rend. Acc. Sci. Torino*. (33): 1002-23.
- Tonelli, Leonida. 1907, Sulle funzioni derivate. *Rend. R. Ist. Lomb. Sci. Lett. II, Vol. XL (Opere di Tonelli 1960, U.M.I. Vol. I, 45-51)*
- Tonelli, Leonida. 1908. Discontinuità di 1^a specie e gruppi di punti. *Rend. R. Ist. Lomb. Sci. Lett. II, Vol. XLI. 773-78 (Anche in Opere di Tonelli 1960, U.M.I., Vol. I, 69-74)*
- Volterra, Vito. 1881a. Alcune osservazioni sulle funzioni punteggiate discontinue, *Giorn. Mat.* (19): 76-86.
- Volterra, Vito. 1881b. Sui principi del calcolo integrale, *Giorn. Mat.* (19): 333-72.
- Volterra, Vito. 1887. Sopra le funzioni che dipendono da altre funzioni. *Rend. Acc. dei Lincei, serie IV, (3): 97-105; 141-146; 153-158.*