transversal
International Journal for the Historiography of Science

Transversal: International Journal for the Historiography of Science, 2024 (17): 1-16
ISSN 2526-2270
Belo Horizonte – MG / Brazil

Article

Leonard Euler in the "Scientific Revolution" Concept of Thomas Kuhn

Dmitri Starostin[1] [https://orcid.org/0000-0001-9448-834X]

Abstract:
This study seeks to clarify T. Kuhn's "theory of scientific revolution" in the application of the laws of I. Newton to the theory of the Moon's motion. It argues that I. Newton's formulae produced at least three approaches. L. Euler provided one by gearing the formula to the needs of practical astronomy and developed calculation techniques that could answer all practical needs, using the apparatus of differentials and derivatives developed by G. W. Leibniz. J. L. Lagrange took Newton's concept of 'force' and developed it into the formalism of 'impulse' and 'energy'. Later G. Plana returned to the point where L. Euler had started, but he used the law of universal gravitation instead of the 2nd law of Newton in his equations. Thus I. Newton's laws and ideas were developed in totally different directions, none of which seemed enough in the 19th century.

Keywords: The Theory of Scientific Revolutions; I. Newton, L. Euler; G. W. Leibniz; J. L. Lagrange; G. Plana; C. Maclaurin; Moon theory; Calculus; Computational Methods.

Received: October 25, 2024. Reviewed: November 10, 2024. Accepted: 2024.
DOI: http://dx.doi.org/10.24117/2526-2270.2024.i17.11

Thomas Kuhn's "Theory of Scientific Revolutions" and A Possibility to Apply It to Leonard Euler's "Moon Theory"

Thomas Kuhn's theory of "Scientific revolutions" provided an influential background to the historiography of scientific advances in early modern Europe. Two publications of this author, one dedicated to the difficulties of the spread of Copernicus' heliocentrical theory and the other to the structure of scientific revolutions, have created the epistemological foundation for discussing progress in science.

In his first monograph, dedicated to the emergence and spread of Nicolas Copernicus' theory, he sought to explain why it took time up to the middle of the 18th century for the evidently correct concept of a heliocentrical system to be accepted universally. As Alexis Clairault still operated in his astronomical calculations of the Moon's motions within the framework of geocentric theory, it was only Leonard Euler who started his work with heliocentric theory as the foundation of his analysis. T. Kuhn's explanation considered the persistence of Ptolemy's astronomical system as a result of its observational veracity (Kuhn 1957).

In the second monograph, he further developed his ideas, suggesting that critical

[1] Dmitri Starostin is an Assistant Professor in the Institute of History at Saint-Petersburg State University. Address: Mendeleevskaia line, 5, Saint-Petersburg 199034, Russian Federation. E-mails: d.starostin@spbu.ru; starostin.dmitry.03@gmail.com

breakthroughs in science start with academic works that were not commonly accepted by the academic community and that were written by individuals whose contribution to scholarship was normally contested for a long time (Kuhn 1962). In other words, T. Kuhn created a consistent concept that explained why the common representations of how the universe functions took so long to be accepted.

Historians of science have recently asked whether T. Kuhn may be considered a classic and whether his heritage is still important as a model theory that can help deploy research in this field (Condé 2023, 16). They also asked whether there is still a possibility to put up an argument with his vision of the Scientific Revolution (Condé 2023, 17). It has been argued that his approach may help researchers advance the history of science from an epistemological perspective (Condé 2023, 20). It was suggested that the comparisons of T. Kuhn with Karl Popper or the representatives of the "New Sociology of Science" movement in academia are largely very superficial since the former sought to find the epistemological basis in constructing a philosophy of science (Condé 2023, 21–23).

This article may specifically be used to add that if the criteria for analysis of advances in science are taken with a view towards the actual algorithms of scientists' exposition of their concepts, the theory of T. Kuhn not only needs no apologetical explanation but retains its revolutionary character and can help find and formulate the vision of critical breakthroughs in the advance of sciences in the 17th and 18th century.

One may agree with his understanding that in the sciences that address the mechanics of the universe, the mainstream accepted concept had become a convenient tradition after several centuries of coming into existence and that it had transformed into a religious system because it could explain astronomical phenomena in a non-contradictory way. His emphasis that it took a considerable amount of time for the breakthrough works, which modern scientists hold as basic, to permeate the study of science can also be taken as a working model.

It seems, however, that his concepts were focused too much on Nicholas Copernicus and Isaac Newton as key figures of a revolution in physics and astronomy and on the difficult path their theories had to tread on their way to universal acceptance. One needs to take into account that T. Kuhn focused only on the development, in the general course of the scientific revolution, of the principles one may call as "metaphysical". On the other hand, I argue that if the criteria of what constituted an advance in physics and mathematics are defined correctly, it becomes clear that T. Kuhn helped create a constructive model for pinpointing the crucial breakthroughs that made the scientific revolution. On the other hand, I hope to suggest that, from the point of view of physics and mathematics, T. Kuhn's terminology and epistemology need to be updated because he was dealing not so much with the practical applications of the physical theory but rather with their metaphysical interpretations. I argue that one needs to take into account the expression of these general statements and laws in a peculiar mathematical form and the deepening of the philosophical concept of motion in light of the challenges posed by the encapsulation of physical phenomena in mathematical formulae of developing calculus.

Thus, to consider the progress in mechanics, one needs to take into account the contributions of other scholars in early modern Europe and, therefore, abstain from imagining them as passive recipients of new concepts and ideas. It is necessary, instead, to investigate the not-so-subtle contributions they had made to modern physics by seeking to find non-contradictory mathematical representations of the laws of Newton that would be deeply rooted in the metaphysical foundations of mathematics. These "elaborations" and "clarifications" were developed with a view to the general principles of science and, I argue, were of nearly the same significance because they covered the practical mathematical representation of the general principles.

My purpose in this article is to suggest that the process of acceptance of a new theory was more complex than scholars gradually agreeing to a generic representation of the new principle. In particular, I believe it is important to add to the discussion within the framework of T. Kuhn's theory of the scientific revolution of the early modern age, the contributions to the development of the laws of I. Newton and J. Kepler which were made by G. W. Leibniz and L. Euler by applying the methods of then-developing calculus to the problems of dynamic motion and, in the case of the latter, to the theory of lunar motion.

In fact, it is necessary to pay particular attention to the ways in which these two latter scholars contributed to building the philosophical foundation of the bridge that was to bring together calculus and dynamics. In particular, one needs to take notice of the fact that the development of critical scientific concepts and representations by such scholars as I. Newton, whom we now deem as towering above others in their field, had significant input from mathematicians like G. W. Leibniz, in the field of calculus, and L. Euler, in the field of formulating the "the second law of Newton" by way of practical application of that calculus to practical problems of astronomy.

In other words, the partly triumphal and partly ironic narrative of the discovery of the laws of dynamics, created by T. Kuhn, needs to be refocused to include sideline stories of equal significance to the general line of development of modern concepts in science. Moreover, it may be relevant to note that in the field of scientific discovery, a bifurcation took place in a field of science, dividing it into at least several sub-processes. In the following discussion, I will investigate how scholars approached I. Newton's ideas from several angles, thus creating their various mathematical representations and apparatuses.

In this article, I seek to investigate how the development of modern astronomical methods in the works of the famous Swiss mathematician and academician of the St. Petersburg Academy of Sciences, Leonard Euler (1707–1783), measures up against the theory of scientific revolutions, advanced in the works of Thomas Kuhn. I will try to prove that Euler's case was special, and that there is no theoretical explanation for it in the theory of T. Kuhn, despite its fundamental nature and heuristic proof.

Taking into account the new approaches to L. Euler's works in the field of the theory of the motion of the Moon, I will try to show that the work of this scientist fell through the cracks of the approach to paradigm shift in science advanced by the abovementioned modern scholar. In showing that L. Euler's works represented an interesting case of balance between purely applied and theoretical science, I will suggest that they had every chance of turning from novel breakthrough mathematical solutions of the Moon's motions' equations to paradigmatic shifts (within the framework of T. Kuhn's theory), if they would not have been eclipsed, not by better mathematical theories of the Moon's motion, but by the breakthrough, sound mathematical apparatus of mathematical analysis and differential calculus, developed by A.-L. Cauchy and K. T. W. Weierstrass.

In light of the new developments in the latter fields, the breakthrough equations by L. Euler, designed to give the correct positions of the Moon, came to be viewed as partial solutions for a limited number of cases rather than a self-sufficient set of methods. It was the contribution of L. Euler, who combined knowledge of the basic principles of physics and was unrivaled in the field of mathematics (and, accordingly, in the field of developing computational methods), that T. Kuhn did not take into account.

Leonard Euler's Contribution to the "Scientific Revolution"

Central to T. Kuhn's theory is the idea of how an established paradigm is subverted by breakthrough works that advance the knowledge in a particular field but that also significantly differ in their methods and apparatus from the established theory. Ptolemy's theory was imprecise and approximate from the modern point of view, but it had long kept its prestige because it had the traits of a religious system. It was because of these faith-related aspects that the heliocentric theory of N. Copernicus could not be accepted for a long time. And it took long for the scientific community universally to accept the ideas of I. Newton who used the theory of the former to make a breakthrough in formulating the law of universal gravitation and the 3 fundamental laws of dynamics (Kuhn 1957, 1962). However, the contribution of Leonard Euler to mathematics, astronomy, and physics has remained outside of T. Kuhn's paradigm. The mathematical advances of Leonard Euler in the field of the Moon's motion have recently received attention from scholars (Verdun 2011, 2013a, 2013b, 2015). This is a welcome addition to the works of specialists in the history of mathematics who have addressed the importance of his heritage for the history of mathematics (Sandifier 2015, 2007; Calinger 2007; Truesdell 1984). This situation in historiography thus allows to put an interesting research question: how can one place Leonard Euler's Moon theories in the context of the theory of a scientific revolution and the paradigm shift that were so aptly advanced by T. Kuhn?

The theory of T. Kuhn gets its first verification check if we address Alexis Clairault and his "Lunar Theory". He started his publications by addressing the problem that had faced astronomers since Hypparchus, the problem of the Earth's precession against the fixed stars (Clairault 1741; Clairaut 1759). His contribution to the problem of Lunar motion lay in putting all astronomical data of the Moon's motion on the solid foundation of Newton's law of gravitation (Clairault 1745). His achievements were best expressed in the article he had been solicited by L. Euler to submit to the Academy of Sciences in St. Petersburg, Russia, the article that got the prize in 1752 (Clairault 1752b). One needs to be aware of the fact that Clairault's belief in the geocentric system naturally undermined the mathematical validity of his calculations (Clairaut [1754]1759). Thus, Alexis Clairault is an example of a scholar who, although accepting a part of I. Newton's dynamics, did not make the final step to making heliocentric theory the foundation of his work. In addition, without using the heliocentric theory, T. Mayer made significant contributions to introducing the apsidial precession of the Moon as a phenomenon, which was shown to have a significant influence on L. Euler (Mayer 1750).

In contrast, L. Euler can be credited with starting all discussions of the Moon's motion only after setting it on a solid foundation of the heliocentric theory. First, let us pay attention to the place that the problem of the Moon's motion occupied in the works of L. Euler. As it has been shown by researchers, this scientist's articles and books on this topic constitute a significant part of his legacy (Verdun 2015). If one wants to understand how L. Euler developed his Moon theory, he or she needs to consider that he counts 3 treatises to him name. The first monograph came out in 1753 as "Theoria motus lunae" (Euler 1753). It appeared after an attempt by A. Clairault to apply I. Newton's law universal gravity to the equations of the Moon's motion emerged as a success (Clairault 1752a, 1752b). He constructed the equation for the Moon's orbit (Clairault 1752b, 57–58). His shortcoming, however, was in him believing in the geocentric system.

So, the book of L. Euler had a significant advantage over his colleague because he put in the foundation of his theory the heliocentrical system. It is appropriate, therefore, to remember here, the theory of scientific progress of T. Kuhn and mark the year 1753 as the turnaround in the acceptance of N. Copernicus' theory as part of the mechanics' mathematical apparatus and starting point. Therefore, L. Euler needs to be credited as the first mathematician who bridged the gap between the theory of universal gravitation and the heliocentric theory on the solid foundation of G. W. Leibniz's calculus. Before his *magnum opus* of 1773, he wrote a critically important article of [1764] 1766, where he critically improved the mathematical apparatus for heliocentrical theory in the latter's application to the motion of the Moon.

He showed that for the Moon to always stay on the line between the Sun and the Earth, it had to be 4 times more distant from the Earth than it was (Euler 1766, 1762, 2022). It was also a breakthrough in the development of theoretical mechanics because the motion of the Moon was considered as a particular case of the three-body problem (Sun-Earth-Moon). This work is relatively underappreciated as the synthesis of the heliocentric theory and the theory of universal gravitation, but it must play a key role as a part of telling the story of the scientific revolution according to T. Kuhn. The final theory of the of the Moon's motion came out in 1773 as "Theoria motuum lunae" (Euler 1772). This work came out as the peak of L. Euler's genius, because he furthered the heliocentric theory by considering the motion of the Moon in three planes against the position of the Earth: The Moon's orbit nodal plane against the plane of the ecliptic, the Moon's apsidal motion plane, and the Moon's orbit precession plane. In these works, Euler's earlier articles were taken account of (Euler 1750, 1746). Let us notice that L. Eulers took the commanding height in providing the mathematical background to the N. Copernicus' and I. Newton's theories and his works may be considered the final point of the acceptance of their theories by the academic community.

This suggests that astronomical problems, and particularly the Moon theory, were central to his research interests. To understand both practical and philosophical contribution of G. W. Leibniz and L. Euler to the problems of dynamics and of lunar motion, one needs to consider several important aspects of their works on calculus (in the case of the former) and on the motion of the Moon (in the case of the latter). In this paper, I propose to analyze passages from one calculus-related manuscript of G. W. Leibniz and one of L. Euler's lesser known works, in which they laid the foundation for appropriately bridging the gap between calculus and dynamics in creating the theory of the Moon's motion from a general foundation of mechanics, the 2nd law of Newton (Euler [1747]1749). In the case of the latter, I intend to look at this work as an example of how the synthesis of mathematics and mechanics made it possible for L. Euler to construct a consistent exposition of mechanics with the help of a logically clear principles of calculus. I will show that critically important for understanding how the 2nd Law of Newton was formulated were the notions of differential and derivative that had been earlier developed by G. W. Leibniz and used in practice of astronomy by L. Euler.

Let us note the form in which he wrote the 2nd Law of I. Newton (Euler [1747]1749, 103).

$$\frac{ddx}{dt^2} = \frac{F_x}{M}; \frac{ddy}{dt^2} = \frac{F_y}{M}; \frac{ddz}{dt^2} = \frac{F_z}{M}$$

This mathematical expression was a significant step ahead of the metaphysical verbal formulation of the law by I. Newton himself: "Mutationem motus proportionalem esse si vi motrici impressae..." (Newton 1686, 12).

Having transformed these expressions into the spherical coordinates, L. Euler reached two following equations: (Euler [1747]1749, 110–111):

I. $2dr\Phi. + rdd\Phi = 0$ (1)

II. $ddr - rd\Phi^2 = \frac{\alpha a^3}{r^2} d\omega^2$ (2)

These two expressions were a step ahead of those found in the works of A. Clairault: $rdv^2 - ddr = \left(\frac{M}{r^2+\Phi}\right) dx^2$(Clairault 1752a, 437). (3)

Let us note that there is no temporal member in the first equation, while it is present in the second equation only as a representation of the objectively present in the perceptions rotation of the Earth. We may, therefore, reject the critical assessment of L. Euler's Moon theories by F. Tisserand, who held the presence of the temporal member as the main shortcoming of the Swiss mathematician's approach (Tisserand 1894, 87–88).

In both equations, we may find a mathematical form of the 2nd Law of J. Kepler: the variation of the square, which is delineated by the lines representing the motion of the celestial body at equal amounts of time, is the same. In other words, a key achievement of L. Euler in this case was using the calculus of G. W. Leibniz to deduce the equations of the motion of the Moon from the 2nd Law of I. Newton and from the 2nd Law of J. Kepler.

G. W. Leibniz, L. Euler, and the Metaphysics of Calculus as Applied to I. Newton's 2nd Law

To understand how much philosophical complexity L. Euler put when applying the calculus of G. W. Leibniz to the 2nd Law of I. Newton, one needs to look at the former's formulation: $\frac{ddx}{dt^2} = \frac{F}{M}$. This was not the traditional $F = ma$ representation that is now firmly associated with Newton's 2nd law. By looking at just this one very small fragment from the work of L. Euler, one may argue that the general principles Newton had put forth in his Laws, the laws of dynamics, never became the point of contention among scholars. In fact, they were universally accepted, which is easily visible if one undertakes to transform the expressions into the now-conventional form. The difference here was in their format since L. Euler, as a practicing astronomer, needed a different formulation of the same principle so that he would be able to deploy them in further investigations correctly.

This interpretation of the key concept of physics, like speed and acceleration needs to be considered as a philosophical breakthrough of G. W. Leibniz and L. Euler, as the discussion had turned away from whether these laws are correct to the problem of their mathematical representation. The starting point for the discussion became the representation of speed as the first derivative of the coordinate function and therefore, of how one can calculate speed and acceleration. It is important to know that Isaac Newton's own formulation of such concepts as speed as the first derivative and of the acceleration as the first derivative of speed in the *Principia* were made in verbal form much as those of Aristotle and his multiple commentators and followers over 2000 years, including Galileo. They later came to be developed as the concept "fluxions" in the works of C. Maclaurin (Maclaurin 1801, 166–202). Although this representation of motion was correct in terms of physics, the problem emerged with mathematically formalizing the representation of acceleration that had only been expressed in verbal form in the *Principia*.

There emerged, as it has been argued, a controversy between I. Newton and G. W. Leibniz about the priority in the formulation of the principles and formulae denoting differentials and derivatives, which were to represent the velocity and acceleration (as the first and the second derivatives). The latter's last years were, as scholars argue, "embittered by a long controversy" between himself, J. Keill and I. Newton about the priority in "inventing" calculus (Grattan-Guinness 1997, 247). This conflict took an extraordinary amount of attention in the academic circles of the late 17th century and exacted a heavy toll on G. W. Leibniz, whose contribution to the problem was long questioned after the first spates of disagreements.

However, the fact that the manuscript of G. W. Leibniz was known to I. Newton makes modern scholars downplay the acrimony of intellectual conflict and reduce it to flashes of hearsay in the context of academic rivalry. C. Maclaurin developed Newton's "dotted" representation of speed and acceleration as derivatives in his "theory of fluxions". However, one of the philosophical problems that emerged out of their expositions was the problem of conceptualizing differential and derivative. The arguments of I. Newton himself and of C. Maclaurin was made on the presumption that the increments of the physical coordinate of a moving body or of its speed were finite ones (Maclaurin 1801, 8, §502). Both were also flouting the idea that acceleration or retardation could be expressed by infinitesimals, but neither of them put forth any formulae to relate their principle (Maclaurin 1801, 2, §495).

Discussing "fluxions", C. Maclaurin constructed a consistent theory, which had as its shortcoming its special terminology that did not aid in understanding his achievement. Thus, using the term "fluxion" mostly to denote what modern scholars call "derivative", and the term "difference" to describe the modern "differential", he sometimes gave explanations without formulae that did not allow for the reader easily to distinguish between the two concepts (Maclaurin 1801, 166–202). The idea that infinitesimally small variation in the parameters of motion could be discarded was constructive in principle (after all, it became the presumption for constructing the Euler-Maclaurin formula), but largely inadequate in the detail of mathematical notation (Maclaurin 1801, 2, §495). From a modern perspective, these vague statements of infinitesimals can be viewed as confusing.

It was G. W. Leibniz who is now usually credited with developing in his manuscripts and published works the modern notation for differential and derivative that lie in the foundation of modern dynamics and of the laws of I. Newton and J. Kepler. Scientists have gotten accustomed to $\frac{dx}{dt}$ for the first derivative and $\frac{d^2x}{dt^2}$ for the 2nd derivative, i.e., for accelleration and so on. Let us note that these simple formulae were much better at conveying the 2nd Law of Newton than his own or C. Maclaurin's expositions. These two scholars' representations presumed that distance, speed, and acceleration were measured only as the function of time. G. W. Leibniz's notation reminded us about that, which was very important for practical astronomical tasks.

The problems of describing physical laws in formulae were of much more profound nature since they required new mathematical conceptualization of such complex ideas as velocity and acceleration. Ever since Zeno came up with his "Aporiae", the concept of velocity and of how to calculate the path of the moving body needed complex philosophical apparatus to be understood. G. W. Leibniz contributed to finding a philosophical foundation to the problem of the finite motion, raising the question of dealing with infinity, the problem that had earlier been addressed by Gregoire de Saint-Vincent and Marin Mersenne (Gregoire de Saint-Vincent 1647; Mersenne 1644b, 1644c, 1644a; Nachtomy 2014; Knobloch 2018; Jesseph 2015; Levey 2015). The case with the acceleration was even more difficult. Scholars of the 17th and 18th centuries had to formulate their ideas of differentiating the functions representing motion before the analysis of the infinitisimally small values was developed in its present form in the fundamental textbook by A. Cauchy (Cauchy 1899, 19).

But G. W. Leibniz's way of creating a mathematical representation of basic physical laws was both a breakthrough and simultaneously fraught with danger of becoming imprecise and approximate because it was wrong to interpret both d^2x and dt^2 as simply the squares of the respective variables. Unlike dx, the expressions d^2t and dt^2 were not implied to be "differentials", as modern mathematicians and physicists know so well. But they could harken back to the expositions of I. Newton and C. Maclaurin. In a sense, therefore, G. W. Leibniz seemed wanting to maintain continuity with these two scholars' and their followers' concepts. But it was key to understand d^2x and dt^2 as the the result of two successful differentiations. The difference between understanding them as symbolic representation of two differentiations and as squares of differentiation lies in the fundamental distinction between the infinitesimally small increment in the value of the variable and its finite increment. The problems of formulating the fundamental laws of physics had a mirror reflection in the difficulties of developing the adequate mathematical notation.

His critical advance over I. Newton and C. Maclaurin was in clearly illuminating what was the parameter under investigation and what was it a function of. Moreover, his notation allowed to avoid uncertainties common to C. Maclaurin's text in distinguishing the "difference" of the fluxion from the "difference" of the "difference" of the fluxion. The formula $\frac{d^2x}{dt^2}$ helped to visualize the latter in a much better way.

It is necessary to take into account that before the breakthroughs of C. Weierstrass and A. Cauchy in the field of mathematical analysis, mathematicians employed different concepts of the differential and derivative that are in use today. For them $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ were not just a representation of a derivative, but an actual result of division of differentials (Cajori 1929, 204). Later, dx came to be considered to be independent of x itself and became to be treated as a simple factor in multiplication (Smirnov 1951a, 123). This understanding did not come to mathematicians immediately, however, since A. Cauchy, in the 1820s, still paid no attention to this fact, even though he approached the theme of parametric functions very closely (Cauchy 1820, 22–26). Remarkably, the key aspect of a mathematician’s skill was to keep in mind that this formula made sense only if x was a variable, independent of y. But the formula for the first derivative $\frac{dy}{dx}$ could be used even if x was not an independent variable (Smirnov 1951a, 172). The first derivative was the ratio of an infinitesimal change in y relative to the infinitesimal change in x. In other words, by making such a formula, Euler specifically hinted that t, time, in his case, was an independent variable, a variable that was not related to the motion of a body. Interestingly, this was a clear mathematical rendering of the First law of Newton, that postulated independent time as the basic parameter of the universe. The search for infinitesimally small values and their representations led Euler and Cauchy to operate with entities like $1-\frac{1}{1+x}$ (Cauchy 1821, 442).

These studies focused on the ways in which the basic principles of dynamics were formulated in the context of the competition between Newton, Leibniz and other scholars who followed them. One needs to pay attention to the fact, however, that the competition needs not be looked for in the general formulations of the laws of physics and their mathematical representations, but also in their specific practical uses. Thus, it is necessary to take special attention when addressing the formula for calculating the motion of the Moon in Euler’s work no. Enestrom 112 (Euler [1747]1749). It is remarkable that to represent the acceleration, the second derivative of the coordinate, Euler chose to write ddx, and not d^2x. This showed that he was well aware of the problem that G. W. Leibniz had encountered: multiplication of differentials led only to power functions of an increasing power, which had a philosophical inconvenience of the $n-th$ derivative becoming $\frac{1}{n}$, which brought back the ghost of Zeno’s “Aporia”. One may remember in this context, that the formula for the 2nd derivative proposed by G. W. Leibniz, the formula, which is most commonly used now, looks in a different way: $\frac{d^2x}{dt^2}$.

The key element of Leibniz’s representation of the derivative may have been construed as taking the second differential (or, in other words, the differential of the differential), as if it were a square of the first differential. His manuscript makes us think that was a possibility for him. As a mathematician, G. W. Leibniz certainly understood that this representation was true only within the framework of certain conditions. But the formula may have reminded those using it of the key computational practice lying under the calculation of differentials and derivatives: the (desired) approximation of any function by polynomials. Nevertheless, the level of the development of mathematics in its relationship to the problems of physics and dynamics in particular was such that the understanding of motion was still being developed and even the best mathematicians needed approximations to capture and conceptualize the key physics’ concepts in their relationship to mathematics. This representation ($\frac{d^2x}{dt^2}$), even though scholars understood its symbolic character as just a type of «notation», made mathematicians conceptualize the second derivative as something they had already been well accustomed with. His notation became especially critical for the formulation of the laws of motion that had, in addition to their metaphysical value, the practical purpose of explaining the motion of the Moon and thus were seminal for calendric and navigational purposes.

The development of Leonard Euler’ variational calculus has been studied in several important articles (Fraser 1989, 1994, 2003). Euler’s importance for development of analytical mechanics in the context of the “mathematization” of dynamics was investigated in the number of works (Panza 1992, 2002). His role in the development of the calculus of the infinitesimally small was studied in several works (Panza 2003, 2005). The framework of the development of the principle of the "least action" of Maupertuis has received its attention in a number of studies (Panza 1995). The problem of the motion of the body in the field of central forces was addressed by using the example of Johann Bernoulli, but not Leonard Euler (Guicciardini 1995). But I seek to prove that L. Euler's work E112 not only shows that by 1750s Leibniz's concept of derivatives took precedence over the dotted "fluxions" of I. Newton and C. Maclaurin, but suggests the Swiss mathematician's significant advance in developing the mathematical apparatus for the second law of Newton (Euler [1747]1749).

It was L. Euler, I argue, who constructed a clear understanding of the difference between an increment, a differential, and a derivative. In fact, I suggest that his role as a mathematical "purist" needs to be addressed within the concept of the "Scientific revolution", by T. Kuhn. I also propose that this treatise needs to be elevated in importance to become one of the key works in the history of the mathematical formulation of the 2nd law of Newton. By the time of J. Euler, the difference between the differential and derivative hat taken hold, since he, as a practicing astronomer, needed clear algorithms which he would deploy in his calculations.

But what philosophical foundations were required to make a philosophically sound concept of speed and acceleration as the first and second derivative? The manuscripts of G. W. Leibniz illustrate that to overcome the "aporiae" of Zeno, he had to construct important concepts that are, speaking in modern terms, related to integrability of a function and that came to constitute the philosophical foundation of dynamics, in general. In particular, one of the questions he posed was that of finding a function that would become itself if differentiated (Leibniz, n.d.). This kind of problem originated in an attempt to measure speed if the acceleration was known, or to measure distance if the speed was known. In the more general terms, this problem originated in seeking to find the distance that an accelerating body would pass in a given amount of time. These kinds of problems emerged from ballistics in the first place, and only partially, from astronomy. This was necessary to describe falling bodies or, in a more practical sense, to determine and calculate paths of bullets and shells. In other words, he was looking for a way to construct the conditions for the exponential function that was later developed in the works of L. Euler.

However, a paradoxical situation has arisen in the formulae of the mathematical theory of the Moon's motion. It has been argued, for example, that Euler was the key scholar in developing the principles pronounced by I. Newton into the formula $F = ma$ that was key to unravel the complex laws of motion for the general public (Truesdell 1984, p. 317; Sitko and da Silva 2021). Thus, it was L. Euler who brought I. Newton's 2nd law into the form in which it is now known, F = ma (Sitko and da Silva 2021, 160; Sitko 2019b, 2019a). Interestingly, while this formula has now gained a common place in the textbooks, L. Euler did not consider it in its present form as a workable model for the start of practical calculations. Instead, he strove to use the parameters that could be obtained by a practical astronomer and that could be employed by those who had studied Ptolemy's methods of using arcs to quantify the movement of the celestial bodies. One needs do keep in mind that the formula of the 2nd law of Newton in its common format F = ma is not quite useful for astronomers and many physicists, since this representation shadowed the important principles that served as a foundation for understanding the motions of physical bodies. For one, this formula implied that with the known mass and acceleration one can calculate the 'force', the entity, the meaning of which may have been imprecise and the value of which might not have had a critically meaningful value in the 17th and 18th centuries. Secondly, one may notice that in this representation of the formula the force is ultimately dependent on the time coordinate. It may suffice for lab experiments, but in the case of celestial bodies and of the law of universal gravitation the force is dependent on the masses and on the distance between bodies. On the one hand, contrary to Euler's suggestion, J. L. Lagrange turned I. Newton's F = ma formula into a different canonical form because he took the latter's 'force' as a starting point in his discussions and coined it into the equation Pdp + Qdq + Rdr = 0 for the motionless state (1st law of I. Newton). This suggested that he was already working with the formalism of modern 'impulse' ($p = mv$) and 'energy' ($E = mv^2 / 2$) parameters (Lagrange, 1811, 28). At the same time, $F = ma$ was more ambiguous in this regard because it shifted the attention of scholars from the acceleration, whether it was constant or variable, to the need to calculate the force, which was not envisioned as a function of time.

But this new formalism was only useful on the theoretical level and could not help much to process astronomical measurements. But there was a strong group among the astronomers like Alexis Clairault who considered the Sun as rotating around the Earth ca. 1752 (Clairault 1752b). Despite the success of the works of J. L. Lagrange, L. Euler's approach was still better suited for the practical needs of astronomy. The formula of Leonard Euler took the force of gravitation and the mass, a set of unknown parameters, out of the group of measurable values and made it into a parameter F/m, or, in the cartesian coordinates, X/m, Y/m, and Z/m (a vector of force divided by the mass). What was left was acceleration, which astronomers could measure directly with their equipment. Moreover, the way Euler presented this formula may have also reminded astronomers that the dependence of a celestial body's coordinate in the sky on time may have been secondary as the gravitational interaction underlay all time coordinates. In other words, in the case of the conjunction of planets or of Jupiter or Saturn being near Earth the periodic motions of the Moon would become perturbed and thus would not be a function of time per se. Leonard Euler knew that and he included the perturbation of the Earth by the Moon in his articles. This meant that one needed to construct a mathematical model better.

This is why in the 19th century P. S. Laplace specifically proposed a competition question to define the motion of the Moon based solely on Newton's law of gravitation, which was won by Giovanni Plana and Francesco Carlini ca. 1832. Unlike L. Euler, who had used the formula for the 2nd law of I. Newton in his form, G. Plana used Newton's law of universal gravitation as his starting point in the form $d^2X / dt^2 = M\, x / r^3 + M'\, x' / r'^3$, the same equation being written out for each coordinate. Let us notice that it is a combination of the 2nd law of Newton and his law of universal gravitation. To account for the fact of the acceleration being itself a function of time, his teachers like Laplace and he himself employed the idea of a perturbation, thus reducing the variable part of the acceleration to a small value. Thus Giovanni Plana later wrote: "The general method by which we will directly determine the three coordinates of the Moon is more suitable to make known the terms which depend on the higher powers of the disturbing force" (Plana, 1832, 189).[2] While this practice temporarily worked for the calculations to achieve a then-required degree of precision, it did not solve the conceptual problem of how to account for the fact that gravitation may be a function of parameters other than time. Giovanni Plana wrote his equations for the motion of the Sun ca. 1832 in such a way that showed him as taking the Earth as the only point of reference. So, he followed Euler's approach in E112 to a maximum, but he did not seem to be employing the apparatus of three-plane reference coordinates in the lattes 1772 book *Theoria motuum lunae.* Thus, the example of Giovanni Plana suggested that the formulation of the second law of I. Netwon by Leonard Euler was geared towards the use of heliocentric system and of the Earth-Moon system as being mutually perturbed.

Thus, in the context of T. Kuhn's approach to scientific revolutions, the situation with the paradigm shift in astronomy looks like the work of at least two great minds, the second of which complemented the initial philosophical turnaround with strict mathematical expressions. But in the tradition of teaching physics, I. Newton has an advantage over L. Euler, although it was the latter who developed his second law into a formula that is widely remembered in the education system. The probable reason was that the most popular edition

[2] "La méthode générale par laquelle nous déterminerons directement les trois coordonnées de la Lune est plus propre à faire connaître les termes qui dépendent des puissances supériores de la force perturbatrice" (Plana 1832, 189).

of Newton's Principia, the so-called "Geneva" one (also known as "Jesuit", 1707) became so popular for the next century after its first publication, that most scholars in Europe learned

$$\int^{a+r\omega} F(a)dx = \omega\left\{\frac{1}{2}F(a) + F(a+\omega) + F(a+2\omega) + \cdots + \frac{1}{2}F(a+r\omega)\right\}$$
$$+\sum_{m=1}^{n-1} \frac{(-1)^m B_m \omega^{2m}}{(2m)!}\left\{F^{(2m-1)}(a+r\omega) - F^{(2m-1)}(a)\right\} + \frac{+\omega^{2n+1}}{(2n)!}\int_0^1 \varphi_{2n}(t)\left\{\sum_{m=0}^{r-1} F^{(2n)}(a+m\omega) + \omega t\right\}$$

about Newton's laws from studying it. It was this textbook that made Newton's verbal formulations and explanations understandable to the reading public (Sitko and da Silva 2021, 165). It is also likely that the reading public did not pay attention to Euler's works in this field because Lagrange's treatise "Analytical Mechanics" did not address his achievements (Sitko and da Silva 2021, 167). Thus, it seems that the innovative work of L. Euler, both in the field of astronomy and the mathematical axiomatics of Newton's second law and Kepler's second law, which provided the opportunity for significant advances in the astronomy of the movement of the Moon, were not fully appreciated in the 18th-19th centuries. They are sometimes assessed inaccurately even by those modern researchers who well understand the internal logic of T. Kuhn's system of ideas. The relevance of our research lies in the fact that we want to point out the primacy of L. Euler precisely in mathematical axiomatics in developing the theory of the motion of the Moon and his attempt to link together, within the framework of one logically developing sequence of formulae, Newton's second law, Kepler's second law, the principle of least action, as well as the concept of "angular momentum", which was still being formed at that time.

Leonard Euler and the "Secular Remainder"

Thus, unlike C. Weierstrass and A. Cauchy, L. Euler transformed the $F = ma$ and its fluxion representation into a formula that presupposed integration. Moreover, using the idea of G. W. Leibniz, that the power function or the reverse power function may help in finding the successful differentials, he effectively proposed that integration of the differential of the visible path of a celestial body may be calculated as a series of reverse power functions. Further development of representations of functions by infinite periodic series by such mathematicians as C. Maclaurin, L. Euler, and J. Stirling had an interesting history, in the details of which one may notice the main problem that L. Euler had in mind, the problem of the periodic motion of the Moon and of a possibility to represent it with a function without any secular members. The expansion of a function into an infinite series that was advanced by J. Stirling and L. Euler at about the same time had as its peculiar possibility to express the function as a periodic series of the variation dx with ever-increasing period of dx plus an additional member (Whittaker and Watson 1920, 7.2.1, 127–128; Korn and Korn 1961, 4.8-10; 1968, 134, 4.8-10; 2000, 125, 4.8-10). In relationship to the Moon's motion, it meant that the latter could be represented as an interposition of time-independent fixed periods (cycles, for example, those of 19 and 95 years) plus a secular remainder that could be reduced to a very small value. In the more general case, the formula looked in the following way:

Let us note this secular remainder in the last section (line) of the formula. The first two members depend only on the products of the ω, the displacement of the main measured coordinate in the form $\omega = (z - a)$, where z is a given value and a is the original displacement. In terms of a Fourier transform, these two members represent the "frequency domain" of the displacement. In other words, these are time-independent characteristics of the cyclical motion of celestial bodies and of the Earth itself as they are visible to the observers. In a sense, this series expansion by Euler and Stirling was a foreshadowing of the Fourier series, since the displacements and their products in the former served like a set of harmonics in the non-secular members of the equation. The secular member and its reduction to very small relative values were the key advancement made by Euler and Stirling in the calculations of the Moon's motion. In the simpler modern notation, this secular member looks in the following way.

$$R_p = (-1)^{(p+1)} \int {}_m^n f^{(p)}(t) \frac{P_p(t)}{p!} d$$

This was the foundation of L. Euler's "philosophy" of calculating the parameters of the Moon's orbit and it allowed to move the precession parameters and the discrepancies between the civic solar calendar and astronomic lunisolar calendar away into a separate equation member. This remainder was philosophically sound from the standpoint of physics and convenient in terms of mathematics. A better methodology of calculating this remainder was further conceived by L. Euler and formalized by B. Riemann in the dzeta-function (Riemann 1859).

The difficulty of imagining motion to be represented by the mathematical identities that had a series of periodic members (represented by a series of a power function) and a part that could not be formalized as such bore down on mathematicians. The meaning of the extra member (R) in the Maclaurin series expansion formula was long misunderstood only as "error". For example, one may find such definition in the classical textbook of the Russian mathematician V. I. Smirnov, who must have been brought up on the approach to calculus of A. Cauchy (Smirnov 1951b, IV.2.128, 309). This scholar, being a "pure mathematician", obviously did not see the applied task of calculating the periods of the motion of the Moon behind the formula of Maclaurin and did not mention L. Euler in conjunction with it.

Conclusion:
Leonard Euler's Heritage in the Light of Thomas Kuhn's Theory

The state of historiography has now achieved its peak in using T. Kuhn's model of the development of the scientific paradigm. Thus, although the accent on the exact participation of scholars in the mathematical formulation of key physical laws have been slightly changed, the major framework remained the same.

Here, I would like to suggest that apart from the question of priority, a much more important problem in the history of early modern physics arose. The direction the discussion went lay in conceptualizing, in the basic terms of Ancient Greek mathematics, the background to philosophical principles of physics. It is in this aspect that we can propose an important addition to T. Kuhn's concept. After the principles (which at first looked more like educated guesses) had been set, contemporary scholars were required to find ways to deploy formulae to describe actual physical or astronomical processes. Once the formulae were deployed, it was equally important to find physical meaning behind these formulae and at each step of their transformations, as well as the mensurability of the parameters employed in them. This set of problems deserves more attention than they have received among the historians of science. The controversy between the I. Newton's and G. W. Leibniz's "camps" allows better investigation along this path of inquiry.

In the more general context, the above considerations let me propose an important addition to the theory of scientific revolution of T. Kuhn. I. Newton's formulae produced at least three approaches. L. Euler provided one by gearing the formula to the needs of practical astronomy that implied the Earth-Moon system on its orbit around the Sun as a starting point of reference. He developed calculation techniques that could answer all practical needs. However, his approach, although correct in the basic physical principles, was not accepted by mathematicians who criticized it for the lack of a universal analytical formula and for being a set of partial solutions. J. L. Lagrange took Newton's concept of 'force' and developed it into the formalism of 'impulse' and 'energy', which had little use for practical astronomy before the breakthroughs of C. G. J. Jacobi and his Jacobian were used in the 20^{th} century to explain the computerized data. Later G. Plana returned to the point where L. Euler had started, but he used the law of universal gravitation instead of the 2^{nd} law of Newton. Because of that he ended up with the formulae with the third power of spatial coordinate in the divider, the formulae which do not allow for an easy solution. Let us thus note that I. Newton's ideas were developed in totally different directions, none of which seemed enough in the 19th century. I make a specific emphasis on the fact that the formulation of the 2nd Law of Newton and of his law of universal gravitation did not cause a doubt or a disagreement with his general idea. But since his second law was formulated in a verbal format, it prompted for an appearance of a series of attempts by mathematicians to construct an ideal mathematical representation that would conform to the stringent requirements of differential and integral calculus. In other words, the theory produced several longstanding traditions of interpreting I. Newton's words as the principle underlying a complex mathematical apparatus. I thus seek to make a clarification to T. Kuhn's theory of the "scientific revolution." It seems that the universal principle that many scholars had accepted at once, was developed in the process of forming a mathematically sound theory of the Moon's motion into several mathematical apparatuses, each with its own set of basic assumptions about key physical notions, measurable parameters and variables.

References

Cajori, F. 1929. *A History of Mathematical Notations.* Vol. 2. Chicago: Open Court.

Calinger, R. S. 2007. "Leonard Euler: Life and Thought." In *Leonard Euler: Life, Work and Legacy,* edited by Sandifer, C. E. and Bradley, R. E., 5–60. Amsterdam ; Boston: Elsevier.

Cauchy, A.-L. 1899. *Cours d'analyse.* Oeuvres complètes d'Augustin Cauchy, series 2 3. Paris: Gauthier-Villars.

Cauchy, A. 1820. "Résume des leçons données à l'école royale polytechnique sur le calcul intinitésimal." In *Oeuvres completés d'Augustin Cauchy, série 2, tome 4,* 9–261.

Cauchy, A. 1821. *Cours d'analyse de l'école royale polytechnique.* Paris: Imprimerie royale.

Clairault, A. 1741. "Sur la manière la plus simple d'examiner si les étoiles fixes ont une parallaxe, et de la déterminer exactement [C24]". *Histoire de l'Academie Royale des Sciences, Mémoires,* 358–369.

Cauchy, A. 1745. "Du système du monde dans les principes de la gravitation universelle [C33]". *Histoire de l'Academie Royale des Sciences, Mémoires,* 329–364.

Cauchy, A. 1752a. "De l'orbite de la Lune, en ne négligeant pas les quarrés des quantités de même ordre que les forces perturbatrices [C40]". *Histoire de l'Academie Royale des sciences, Mémoires,* 421–440.

Cauchy, A. 1752b. *Théorie de la Lune déduite du seul principe de l'attraction réciproquement proportionelle (sic) aux quarrés des distances... Pièce qui a remporté le prix de l'Académie impériale des sciences de Saint Pétersbourg en 1750 [C39].* Saint-Pétersbourg: Academie Imperiale de Sint-Petersburg.

Clairaut, A. 1759. "Einfache Art, zu untersuchen, ob die Fixsterne eine Parallaxe haben, und sie genau zu bestimmen [C. 24A]". *Der Königl. Akademie der Wissenschaften in Paris physische Abhandlungen* (13): 116–131.

Clairaut, A. [1754]1759. "Mémoire sur l'orbite apparente du Soleil autour de la Terre, en ayant garde aux perturbations produites par les actions de la Lune et des planètes principales [C47]." *Histoire de l'Academie Royale des Sciences. Mémoire,* 521–564.

Condé, M. L. 2023. "A herança de Thomas Kuhn para a história e a filosofia da ciência ["Thomas Kuhn's Heritage for the History and Philosophy of Science"]. *Problemata*: *Revista Internacional de Filosofia* 14 (4): 15–26.

Euler, L. 1746. "De motu nodorum lunae ejusque inclinationis ad eclipticam variation [E138a]". *Histoire de l'academie des sciences de Berlin* 1:40–44.

Euler, L. 1750. "De motu nodorum lunae ejusque inclinationis ad eclipticam variation [E138]." *Novi commentarii academiae scientiarum Petropolitanae* 1:387–427.

Euler, L. 1753. *Theoria motus lunae exhibens omnes ejus inaequalitates In additamento hoc idem* *argumentum aliter tractatur simulque ostenditur quemadmodum motus lunae cum omnibus inaequalitatibus innumeris aliis modis repraesentari atque ad calculum revocari possit auctore L. Eulero Impensis academiae imperialis scientiarum Petropolitanae anno 1753 [E187]*. St.Petersbourg: Academie des Sciences de St.Petersbourg.

Euler, L. 1766. "Considerationes de motu corporum coelestium (Presented to Berlin Academy on April 22, 1762 and to St.Petersburg Academy on May 17, 1762) [E304]". *Novi Commentarii academiae scientiarum Petropolitanae* 10:544–558.

Euler, L. 1772. *Theoria motuum lunae, novo metodo pertractata una cum tabulis astronomicis, unde ad quodvis tempus loca lunae expedite computari possunt incredibile studio atque indefesso labore trium academicorum: Johannis Alberti Euler, Wolffgangi Ludovici Krafft, Johannis Andreae Lexell. Opus dirigente Leonardo Eulero acad. scient. Borussicae directore vicennali et socio acad. Petrop. Parisin. et Lond. [E418]*. Petropoli: Typis academiae imperialis scientiarum.

Euler, L. 2022. "Considerations on the motions of the celestial bodies [E304]". Translated by Bistafa, S. R.

Euler, L. 1762. "Considerationes de motu corporum coelestium (Presented to Berlin Academy on April 22, 1762 and to St.Petersburg Academy on May 17, 1762) [E304]". In *Leonardi Euleri Opera Omnia, series 2*, 25:246–257.

Euler, L. [1747]1749. "Recherches sur le mouvement des corps c´elestes en général". *Mémoires de l'academie des sciences de Berlin* 3:93–143.

Fraser, C. G. 1989. "The calculus as algebraic analysis: Some observations on mathematical analysis in the 18th century". *Archive for History of Exact Sciences* (390): 317–335.

Fraser, C. G. 1994. "The origins of Euler's variational calculus". *Archive for History of Exact Sciences* (470): 103–141.

Fraser, C. G. 2003. "The calculus of variations: A historical survey". In *A history of analysis*, edited by Jahnke, H. N., 355–384. New York: American Mathematical Society / London Mathematical Society.

Grattan-Guinness, I. 1997. *The Norton History of the Mathematical Sciences.* London: W. W. Norton.

Gregoire de Saint-Vincent. 1647. *Problema austriacum plus ultra quadratura circuli.* 2 vols. Antverpiae: J. et J. Meursios.

Guicciardini, N. 1995. "Johann Bernoulli, John Keill and the inverse problem of central forces". *Annals of Science* (52): 537–575.

Jesseph, D. M. 2015. "G.W. Leibniz, Interrelations Between Mathematics and Philosophy". Edited by Jesseph, D. M., 189–205. Springer Verlag.

Knobloch, E. 2018. "Leibniz and the Infinite." *Quaderns d'Història d'Enginyeria* (16): 11–31.

Korn, G. T. and Korn, T. M. 1961. *Mathematical Handbook for Scientists and Engineers: Definitions, Theorems and Formulas for Reference and Review.* New York: McGraw-Hill.

Korn, G. T. and Korn, T. M. 1968. *Mathematical Handbook for Scientists and Engineers: Definitions, Theorems and Formulas for Reference and Review (in Russian).* Moscow: Nauka.

Korn, G. T. and Korn, T. M. 2000. *Mathematical Handbook for Scientists and Engineers: Definitions, Theorems and Formulas for Reference and Review.* Mineola, NY: Dover Publications.

Kuhn, T. S. 1957. *The Copernican Revolution: Planetary Astronomy in the Development of Western Thought.* Cambridge: Harvard University Press.

Kuhn, T. S. 1962. *The Structure of Scientific Revolutions.* Chicago: Harvard University Press.

Lagrange J. L. 1811. *Mécanique analytique*. T. 1. Paris, Courcier.

Leibniz, G. W. n.d. *[Calculus in differentialibus replicatus per semisubstitutiones].* Niederšachsische Landesbibliothek, Gottfried Wilhelm Leibniz Bibliothek, Call number: LH 35, 8, 9 Bl. 6-7.

Levey, S. 2015. "G.W. Leibniz, Interrelations Between Mathematics and Philosophy". Edited

by Jesseph, D. M., 157–187. Springer Verlag.
Maclaurin, C. 1801. *A Treatise on Fluxions.* 2nd ed. Vol. 2. London: William Baynes / William Davis.
Mayer, T. 1750. "Abhandlung ̈uber die Umwälzung des Monds um seine Axe, und die scheinbare Bewegung der Mondsflecken, worinnen der Grund einer verbesserten Mondbeschreibung aus neuen Beobachtungen gelegt wird". In *Kosmographische Nachrichten unt Sammlungen auf d. J. 1748, von J. T. Mayer besorgt,* by Mayer, T., 52–183. Nuremberg.
Mersenne, M. 1644a. *De ballistica et acontismologia seu de sagittarum, iaculorum et aliorum missilium iactibus.* Paris: Antonius Bertier.
Mersenne, M. 1644b. *F. Marini Mersenni minimi Cogitata physico-mathematica.* Paris: Antonius Bertier.
Mersenne, M. 1644c. *Hydraulica, pneumatica, arsque navigandi, harmonia theorica et practica. Praefatio ad lectorem.* Paris: Antonius Bertier.
Nachtomy, O. 2014. "Infinity and Life: The Role of Infinity in Leibniz's Theory of Living Beings". In *The Life Sciences in Early Modern Philosophy.* Oxford University Press, January.
Newton, I. 1686. *Philosophiae Naturalis Principia Mathematica.* London: Pepys.
Panza, M. 1992. *La forma della quantità. Analisi algebrica e analisi superiore: Il problema dell'unità della matematica nel secondo dell'illuminismo.* 2 vols. Cahiers d'histoire et de philosophie des sciences. Nouvelle série, 38-39. Paris: Société française d'histoire des sciences.
Panza, M. 1995. "De la nature épargnante aux forces généreuses. Le principe de moindre action entre mathématique et métaphysique: Maupertuis et Euler (1740-1751)". *Revue d'Histoire des Sciences* 48 (4): 435–520.
Panza, M. 2002. "Mathematisation of the science of motion and the birth of analytical mechanics: A historiographical note". In *The Application of Mathematics to the Sciences of Nature. Critical moments and Aspects,* edited by Cerrai, P., Freguglia, P., and Pellegrini, C., 253–271. New York: Kluwer/Plenum publishers.
Panza, M. 2003. "The origins of analytic mechanics in the 18th century". In *A History of Analysis*. Edited by Jahnke, H. N., 137–153. New York: American Mathematical Society / London Mathematical Society.
Panza, M. 2005. *Newton et les origines de l'analyse: 1664-1666.* Paris: Librairie Albert Blanchard.
Plana, G. A. A. 1832. *Théorie du mouvement de la Lune*. Tome 1. Turin, 1832.
Riemann, B. 1859. "Über die Anzahl der Primzahlen unter einer gegebenen Grösse". *Monatsberichte der Berliner Akademie* (November).
Sandifier, C. E. 2007. "Leonhard Euler: life, work and legacy: introduction". In *Leonard Euler: Life, Work and Legacy*. Edited by Sandifer, C. E. and Bradley, R. E., 1–4. Amsterdam; Boston: Elsevier.
Sandifier, C. E. 2015. *How Euler Did Even More.* Washington, DC: Mathematical Association of America.
Sitko, C. M. 2019a. "Os desenvolvimentos da Mecânica Analítica que culminaram na elaboração de $F = ma$". *Caderno Brasileiro de Ensino de Física* 36 (1).
Sitko, C. M. 2019b. "Why Newton's Second Law is not $F = ma$". *Acta Scientiae* 21 (1): 83–94.
Sitko, C. M. and da Silva, M. R. 2021. "Kuhnian Analysis of Why, even after Euler's Contributions, the Fundamental Principle of Motion is still 'Newton's Second Law'". *Acta Scientiae* 21 (3).
Smirnov, V. I. 1951a. *A Course of Higher Mathematics [Kurs vysshei matematiki].* Vol. 1. Moscow: Gosudarstvennoe izdatel'stvo tehniko-teoreticheskoi literatury.
Smirnov, V. I. 1951b. *Kurs vysshei matematiki.* 12th ed. Vol. 1. Moscow, Leningrad: Nauka.
Tisserand, F. 1894. *Traité de mécanique céleste.* Vol. 3. Paris: Gauthier-Villars et fils.
Truesdell, C. A. 1984. *An idiot's fugitive essays on science methods: methods, criticism, training, circumstances.* New York; Berlin; Heidelberg: Springer-Verlag.

Verdun, A. 2011. "Die (Wieder-)Entdeckung von Eulers Mondtafeln – The Discovery of Euler's Lunar Tables." *NTM Zeitschrift für Geschichte der Wissenschaften, Technik und Medizin* 19 (3): 271–297.

Verdun, A. 2013a. "Leonard Euler's early lunar theories 1725-1752, Part 1: first approaches, 1725-1730". *Journal for the History of Exact Sciences* (67): 235–303.

Verdun, A. 2013b. "Leonard Euler's early lunar theories 1725-1752, Part 2: Developing the Methods, 1730-1744". *Journal for the History of Exact Sciences* (67): 477–551.

Verdun, A. 2015. *Leonhard Eulers Arbeiten zur Himmelsmechanik.* Berlin: Springer.

Whittaker, E. T. and Watson, G. N. 1920. *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions, with an Account of the Principal Transcendental Functions.* 4th ed. Vol. 1. Cambridge: Cambridge University Press.